

= 0 = PROBABILITY = 0 =

Sample space :- A set of all possible outcomes of an experiment is called sample space.

Ex :- In throwing a die. There are six possibilities 1, 2, 3, 4, 5 and 6.

∴ The sample space is $S = [1, 2, 3, 4, 5, 6]$.

Ex :- Tossing a coin. There are two possible cases either Head (or) Tail (T).

Random Experiment :-

Probabilistic situation is referred to as a random experiment.

Trial :- Each performance in a random experiment is called a trial. i.e. all the trials conducted under the same set of conditions from a random experiment.

Outcome :-

The result of trial in a random experiment is called an outcome.

Event :- Every non empty subset of a sample space of a random experiment is called an event.

Favourable Cases :-

The number of cases favourable to an event in trial is the number of outcomes which entail the happening of an event.

Ex:- In throwing two dice, The number of cases favourable getting a sum 11
= [5,6] (6,5).

Mutually exclusive :-

Two events A, B of a sample space
are said to be mutually exclusive, if they have no sample points in common i.e. $A \cap B = \emptyset$

* Mutually exclusive events are sometimes called as disjoint events.

(or).

Events are said to be mutually exclusive if the happening of anyone of them precludes the happening of all the others. That is no two or more of them can happen simultaneously in the same trial.

Ex:- In tossing a coin the events Head turning up and Tail turning up are mutually exclusive.

Ex:- In throwing a die all 6-possible cases are mutually exclusive.

Exhaustive events :-

All possible events in an trial are known as exhaustive events.

Equally likely :-

Events are said to be equally likely when there is no reason to expect anyone of them rather than anyone of the others.

(or).

outcomes of a trial are said to be equally likely if one cannot be expected in preference to the others. (2)

Ex:- In tossing a coin two possible cases are equally likely.

Probability :-

If an experiment is performed n is the number of exhaustive cases and m is the number of favourable cases of an event A . Then probability of an event A is defined by

$$\frac{\text{number of Favourable cases}}{\text{number of Exhaustive cases}} = \frac{m}{n} \\ = \frac{n(A)}{n(S)}$$

Where.

$n(A)$ = number of elements belonging to A .

$n(S)$ = number of elements belonging to S [sample space]

Ex:-

Problem ①:- Find the probability of getting a Head in tossing a coin.

Soln:- number of exhaustive cases = 2.

number of possible case = 1

$$\therefore \text{probability} = \frac{1}{2}.$$

② Find the probability of getting a Head in tossing two coins.

Let A be the event of getting one head.

$$A = (\text{HT}, \text{TH})$$

$$S = [\text{HT}, \text{HH}, \text{TH}, \text{TT}]$$

no. of exhaustive cases $S = 4$

no. of possible cases $A = 2$.

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$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}.$$

Diamonds Heart spades



clubs

① Find the probability of getting one head in tossing two coins.

Let A be the event of getting one head.

$$A = [HT, TH]$$

$$S = [HH, HT, TH, TT]$$

no. of exhaustive cases ($S = 4$)

no. of possible cases ($A = 2$)

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

② Find the probability of getting one red king if we select a card from a pack of 52 cards.

There are 2 red kings

\therefore no. of possible cases = 2.

no. of exhaustive cases = 52.

$$\therefore \text{Probability} = \frac{2}{52} = \frac{1}{26}$$

③ If three coins are tossed. Find the probability of getting (i). Three heads.

(ii). Two heads

(iii). no heads.

Solⁿ :- (i). Let the event A be of getting three heads.

$$A = [H H H]$$

$$S = [HHH, HHT, THH, HTT, HTH, THT, TTH, TTT]$$

$$\therefore P(A) = \frac{1}{8}$$

(ii). Let the event B be getting two heads.

$$B = [HHT, THH, HTH]$$

$$P(B) = \frac{3}{8}$$

(iii). no heads.

Let the event C be getting no head.

$$C = [TTT]$$

$$\therefore P(C) = \frac{1}{8}.$$

④ What is the probability of drawing an ace from a well shuffled deck of 52 playing cards?

Soln :- The number of exhaustive cases $= 52C_1 = 52$

$$= \frac{52 \times 51 \times 50!}{(52-1)! \times 1!} \quad nCr = \frac{n!}{(n-r)! r!}$$

$$= \frac{52 \times 51 \times 50!}{51 \times 50!} = 52.$$

The number of possible cases $4C_1$ (since there are 4-cases)=4

$$\therefore \text{Probability} = \frac{\text{no. of possible cases}}{\text{no. of exhaustive cases}}$$

$$= \frac{4}{52} = \frac{1}{13}.$$

⑤ Two cards are selected at random from 10 cards numbered 1 to 10. Find the probability that the sum is even if (i). The two cards are drawn together (ii). The two cards are drawn one after the other with replacement.

Soln :- No. of Exhaustive cases $10C_2 = 45$

The sum is even if both are odd (or) both are even.

There are 5 even and 5 odd cards.

both are even cases $5C_2 = 10$

both one odd cases $5C_1 \times 5C_1 = 25$

$$\therefore \text{No. of possible cases} = 10 + 25 = 35$$

$$\therefore \text{Required Probability} = \frac{35}{45} = \frac{7}{9}.$$

(ii). Two cards are drawn one after the other with replacement.

$$\therefore \text{No. of exhaustive cases} = 10 \times 10 = 100$$

$$\text{No. of possible cases both are even} = 5 \times 5 = 25$$

$$\text{No. of possible cases both are odd} = 5 \times 5 = 25$$

$$\therefore \text{The no. of possible cases} = 25 + 25 = 50.$$

$$\therefore \text{Required probability} = \frac{50}{100} = \frac{1}{2}.$$

⑥ A Bag contains 5 red balls, 8 blue balls and 11 white balls are drawn together from the box. Find the probability that (i). One is red, one is blue and one is white

(ii). Two whites and one red. (iii). Three white.

Soln: No. of exhaustive cases ${}^{24}C_3 = 2024$.

(i). No. of possible cases ${}^5C_1 \cdot {}^8C_1 \cdot {}^{11}C_1 = 440$.

$$\therefore \text{Required Probability} = \frac{\text{no. of possible cases}}{\text{no. of exhaustive cases}}$$
$$= \frac{440}{2024} = \frac{55}{253}.$$

(ii). No. of possible cases ${}^{11}C_2 \cdot {}^5C_1$

$$= 55 \times 5 = 275,$$

$$\therefore \text{Required Probability} = \frac{\text{no. of possible cases}}{\text{no. of exhaustive cases}}$$
$$= \frac{275}{2024} = \frac{25}{184}.$$

(iii). No. of possible cases ${}^{11}C_3 = 165$

$$\therefore \text{Required probability} = \frac{165}{2024} = \frac{15}{184}.$$

⑦ Find the probability of getting a sum 9 if two dice are thrown.

⑧ Three cards are drawn from a pack of 52 cards
find the probability that.

(i). 3 are spades.

(ii). 2 spades one diamond.

(iii). 1 spade, 1 diamond, 1 heart.

* PROBABILITY - AXIOMATIC APPROACH

Def :- Let S - be a finite sample space a real value

function P from the power set of S into R is
called function on S if the following three axioms
are satisfied.

Axioms of Probability

(i). Axiom of Positivity : $P(E) \geq 0$ for every subset
 E of S .

(ii) Axiom of Certainty : $P(S) = 1$

(iii) Axiom of Union : if E_1, E_2 are disjoint subset
of S .
Then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

* The Image $P(E)$ of E is called the probability
of the event E .

Ex :- In tossing two coins sample space.

$$S = [HH, HT, TH, TT]$$

$$P[HH] = \frac{1}{4}$$

$$P[HT, TH] = \frac{1}{2}$$

$$P(TT) = \frac{1}{4}$$

$$P(S) = P(HH) + P[HT, TH] + P(TT) = 1$$

There are mutually exclusive cases.

* SOME ELEMENTARY THEOREMS

Theorem ① :- Probability of complementary event

$$P(A^c) = 1 - P(A).$$

$$S = A \cup A^c$$

A and A^c are mutually exclusive events (or) disjoint.

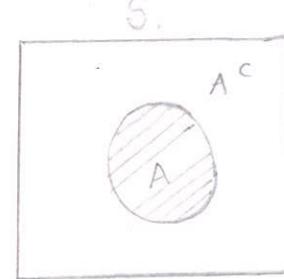
$$\therefore A \cap A^c = \emptyset$$

$$\therefore P(S) = P(A) + P(A^c)$$

$$P(S) = 1$$

$$\therefore 1 = P(A) + P(A^c)$$

$$P(A^c) = 1 - P(A).$$



Theorem ② :- For any two events A and B : $P(A \cap B)$

$$P(A \cap B) = P(B) - P(A \cap B).$$

$A \cap B$ is shown by horizontal lines
 $A^c \cap B$ by vertical lines.

\therefore Events $A \cap B$ and $A^c \cap B$ are mutually exclusive
(or) disjoint

$$\therefore P[A \cap B] \cup [A^c \cap B]$$

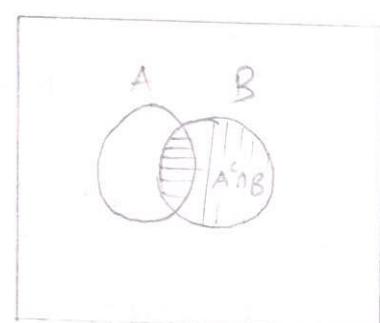
$$= P(A \cap B) + P(A^c \cap B)$$

$$\text{But } (A \cap B) \cup (A^c \cap B) = B$$

$$\therefore P(B) = P(A \cap B) + P(A^c \cap B)$$

$$\therefore P(A^c \cap B) = P(B) - P(A \cap B).$$

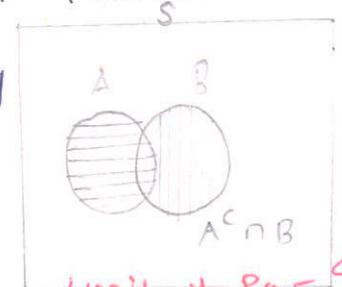
Note :- similarly we can prove $P(A^c \cap B^c) = P(S) - P(A \cap B)$.



Theorem ③ :- ADDITION THEOREM OF PROBABILITY.

If S is sample and A, B are any events in S . Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



Proof :- Set A is shown by horizontal lines

$A^c \cap B$ by vertical lines

A and $A^c \cap B$ are mutually exclusive (or) disjoint events

$$\therefore P[A \cup (A^c \cap B)] = P(A) + P(A^c \cap B)$$

$$A \cup (A^c \cap B) = A \cup B$$

$$\therefore P(A \cup B) = P[A \cup (A^c \cap B)]$$

$$= P(A) + P(A^c \cap B)$$

$$= P(A) + P(B) - P(A \cap B)$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Theorem ④ :- For any three Events A, B and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) + P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) + P(A \cap B \cap C)$$

Proof :- $P(A \cup B \cup C) = P(D \cup C)$ where $D = A \cup B$.

$$P(D \cup C) = P(D) + P(C) - P(D \cap C) \quad \text{--- ①}$$

$$= P(A \cup B) + P(C) - P((A \cup B) \cap C)$$

$$= P(A) + P(B) - P(A \cap B) + P(C) - P[(A \cap C) \cup (B \cap C)] \quad \text{--- ②}$$

$$[\because P(A \cup B) = P(A) + P(B) - P(A \cap B) \text{ &}$$

$$\text{distributive law } (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

$$= P(A) + P(B) - P(A \cap B) + P(C) - P[(A \cap C) + P(B \cap C) - P(A \cap B \cap C)] \quad \text{--- ③}$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \quad \text{--- ④}$$

$$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$

Theorem ⑤: If A and B are independent events, then A^c and B^c are also independent events.

Proof: Given that A and B are independent events

$$\therefore P(A \cap B) = P(A) \cdot P(B) \quad \therefore A^c \cap B^c = (A \cup B)^c - (A \cup B)$$

$$P(A^c \cap B^c) = P((A \cup B)^c) \quad (\text{De Morgan law}) \quad \therefore [A \cap B]^c = (A \cap B)^c \\ = A' \cup B'$$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

$$= 1 - P(A) - P(B) + P(A) \cdot P(B)$$

$$= [1 - P(A)] [1 - P(B)]. \quad [\because P(A \cap B) = P(A) \cdot P(B)]$$

$$= P(A^c) P(B^c)$$

$\therefore A^c$ and B^c are independent.

Theorem ⑥: If A and B are independent events, then A and B^c are independent.

Proof: Given that A and B are independent

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

$$= P(A) - P(A) P(B)$$

$$= P(A) (1 - P(B))$$

$$= P(A) \cdot P(B^c)$$

$$P(A \cap B^c) = P(A) \cdot P(B^c)$$

$\therefore A$ and B^c are independent events

Note:- Similarly we can prove A^c and B are independent.

Problem ①:- A card is drawn from a well shuffled pack of cards what is the probability that is either a spade or an ace?

Soln :- Let S is the sample space of all the sample space
 $\therefore n(S) = 52.$

Let 'A' denote the event of getting a spade and B denotes the event of getting an ace?

Then $A \cup B$ = the event of getting a spade or an ace

$A \cap B$ = The event of getting a spade and an ace.

$$P(A) = \frac{13}{52} \quad P(B) = \frac{4}{52}$$

$$P(A \cap B) = \frac{1}{52}.$$

By addition theorem

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}.$$

② Three students A, B, C are in running race. A and B have the same probability of winning and each is twice as likely to win as C. Find the probability that B or twins.

Soln :- $A \cup B \cup C = S$ = Sample Space

By data $P(A) = P(B)$ and $P(A) = 2P(C)$ - ①

$$\text{We have } P(A) + P(B) + P(C) = 1$$

$$2P(C) + 2P(C) + P(C) = 1$$

$$5P(C) = 1 \Rightarrow P(C) = \frac{1}{5}$$

$$P(A) = \frac{2}{5} \text{ and } P(B) = \frac{2}{5}.$$

The probability that B or C wins = $P(B \cup C)$

$$= P(B) + P(C) - P(B \cap C)$$

$$= \frac{2}{5} + \frac{1}{5} - 0 = \frac{3}{5}.$$

③ Out of 15 items 4 are not in good condition, 4 are selected at random. Find the probability that
 (i). All are not good (ii). Two are not good.

Sol :- Total number of items = 15

number of ways of picking 4 items = ${}^{15}C_4$

(i). Suppose 4 items are chosen which are not good

No. of ways of selecting = 4C_4

\therefore The probability all are not good = $\frac{{}^4C_4}{{}^{15}C_4} = \frac{1}{1365}$

(ii) Suppose two items are not good

No. of ways of selecting of 2 bad items = 4C_2 .

Probability of getting two items which are

$$\begin{aligned}\text{are not good} &= \frac{{}^4C_2}{{}^{15}C_4} = \frac{\frac{4 \times 3 \times 2}{2 \times 2}}{\frac{15 \times 14 \times 13 \times 12 \times 11}{1 \times 4 \times 3 \times 2}} \\ &= \frac{3 \times 2}{15 \times 7 \times 3} = \frac{2}{455}.\end{aligned}$$

Conditional event :- If E_1, E_2 are events of a sample space 'S' and if E_2 occurs after the occurrence of E_1 , then the event of occurrence of E_1 after the event E_1 is called conditional event of E_2 given E_1 . It is denoted by E_2/E_1 .

We define E_2/E_1 .

CONDITIONAL PROBABILITY :-

Def :- If E_1 and E_2 are two events in a sample space 'S' and $P(E_1) \neq 0$, then the probability of E_2 , after the event E_1 has occurred is called conditional probability of the event of E_2 is given -

E_1 and is denoted by $P(E_2/E_1)$ and

$$\text{we define } P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

$$\text{Similarly we define } P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

$$\begin{aligned} \text{We have } P\left(\frac{E_2}{E_1}\right) &= \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{n(E_1 \cap E_2)/n(S)}{n(E_1)/n(S)} = \\ &= \frac{n(E_1 \cap E_2)}{n(E_1)} \\ &= \frac{\text{no.of elements in } E_1 \cap E_2}{\text{no.of elements in } E_1} \end{aligned}$$

Here E_1 is s and $P(E_1) > 0$

Multiplication Theorem :- In a random experiment

If E_1, E_2 are two events

such that $P(E_1) \neq 0$ and $P(E_2) \neq 0$ then the conditional probability E_2 given E_1

$$= P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)} \quad [\because \text{By the multiplication theorem}]$$

$$(i) \Rightarrow P(E_1 \cap E_2) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right)$$

$$(ii) \Rightarrow P(E_2 \cap E_1) = P(E_2) \cdot P\left(\frac{E_1}{E_2}\right).$$

Note :- Extended three events E_1, E_2, E_3 as.

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P\left(\frac{E_1}{E_2}\right) \cdot P\left(\frac{E_2}{E_1 \cap E_3}\right).$$

The result can be extended to 4 (or) more events

THEOREM A BOUT MULTIPLICATION

Problem ① :- A class has 10 boys and 5 girls. Three students are selected random one after another. Find the probability that

- First two are boys and third is girl.
- First and Third are of same Gender and the second is of opposite gender.

Sol :- Total no. of students = $10 + 5 = 15$

(i). The probability that the first two are boys and the third girl is. $P(E_1 \cap E_2 \cap E_3) = \frac{10}{15} \cdot \frac{9}{14} \cdot \frac{5}{13}$
 $= \frac{15}{91} = 0.1648$.

(ii) Suppose the first and third are boys and second is a girl.

$$\text{Probability of the event} = P(E_1) = \frac{10}{15} \cdot \frac{5}{14} \cdot \frac{9}{13}$$
 $= \frac{15}{91}$

Suppose first and third are girls and second is boy

Then the probability of the event = $P(E_2)$.

 $= \frac{5}{15} \cdot \frac{10}{14} \cdot \frac{11}{13}$
 $= \frac{20}{273}$

\therefore Required Probability = $P(E_1) + P(E_2)$

$= \frac{15}{91} + \frac{20}{273}$

$= \frac{45+65}{273} = 0.238$

② Two aeroplanes bomb a tangent target in succession.

The probability of each correctly scoring a hit is 0.3 and 0.2 respectively. The second will bomb only if the first misses the target. Find the probability that

- target is hit.

(ii). both fails to score hits.

SOL :- Let A be the event of first plane hitting

the target and B be the event of 2nd plane

hitting the target

The probability of 1st plane hitting the target

$$P_S = P(A) = 0.3$$

The probability of 2nd plane hitting the target

$$P_S = P(B) = 0.2$$

$$P(\bar{A}) = 1 - P(A)$$

$$= 1 - 0.3 = 0.7 \text{ and}$$

$$P(\bar{B}) = 1 - P(B)$$

$$= 1 - 0.2$$

$$= 0.8.$$

(i). $P(\text{target is hit}) = P[(A \text{ hits}) \text{ or } (A \text{ fails and } B \text{ hits})]$

$$= P(A \cup (\bar{A} \cap \bar{B}))$$

$$= P(A) + P(\bar{A} \cap \bar{B}) \quad [\because \text{Axiom(iii)}]$$

$$= P(A) + P(\bar{A}) \cdot P(B)$$

$$= 0.3 + (0.7)(0.2) = 0.44.$$

(\because my multiplication)

[$\because \bar{A} \& B$ are independent]

(ii). $P(\text{both fails}) = P(A \text{ fails and } B \text{ fails})$

$$= P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$$

$$= (0.7)(0.8) = 0.56.$$

③ Determine (i) $P(B|A)$, (ii) $P\left(\frac{A}{B^c}\right)$, if A and B are

events with $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(A \cup B) = \frac{1}{2}$.

SOL :- Given $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$, $P(A \cup B) = \frac{1}{2}$

Now $P(A \cap B) = P(A) + P(B) - P(A \cup B)$.

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{2}$$

$$= \frac{4+3-6}{12} = \frac{1}{12}$$

$$P(A \cup B) = \frac{1}{12}$$

$$(i). P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{4}$$

$$(ii). P(B^c) = 1 - P(B) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

$$= \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$$

$$P\left(\frac{A}{B^c}\right) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

- (4) Box A contains 5 red and 3 white marbles and Box B contains 4 red and 6 white marbles. If a marble is drawn from each box what is the probability that they are both of same colour.

Soln :- suppose E_1 = The event that the marble is drawn from box A and is red.

$$P(E_1) = \frac{1}{2} \cdot \frac{5}{8} = \frac{5}{16}$$

and E_2 = The event that the marble is from box B and is red

$$\therefore P(E_2) = \frac{1}{2} \cdot \frac{2}{8} = \frac{1}{8}$$

The probability that both the marbles are red is

$$P(E_1 \cap E_2) = P(E_1) P(E_2) = \frac{5}{16} \cdot \frac{1}{8} = \frac{5}{128}$$

Let E_3 = The event that the marble drawn from box A is white.

$$\therefore P(E_3) = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16}$$

Let E_4 = The event that the marble drawn from box B is white

$$\therefore P(E_4) = \frac{1}{2} \cdot \frac{6}{8} = \frac{3}{8}$$

$$\text{and } P(E_3 \cap E_4) = \frac{3}{16} \cdot \frac{3}{8} = \frac{9}{128}$$

The probability that marbles are of same colour

$$= P(E_1 \cap E_2) + P(E_3 \cap E_4)$$

$$= \frac{5}{128} + \frac{9}{128} = \frac{14}{128} = \frac{7}{64} = 0.109.$$

- ② Two marbles are drawn in succession from a box containing 10 red, 30 white, 20 blue and 15 orange marbles with replacement being made after each draw. Find the probability that

(i). Both are white

(ii). First is red and second is white.

Soln :- Total no. of marbles in the box = 75

(i). Let E_1 be the event of the first drawn marble

is white then $P(E_1) = \frac{30}{75}$

Let E_2 be the event of second drawn marble.

is also white then $P(E_2) = \frac{30}{75}$

The probability that both marbles are white

$$= P(E_1 \cap E_2) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right) = \frac{30}{75} \cdot \frac{30}{75} = \frac{4}{25}$$

(ii). Let E_1 be the event that the first drawn marble is red then $P(E_1) = \frac{10}{75} = \frac{2}{15}$

Let E_2 be the event that the drawn marble

is white then $P\left(\frac{E_2}{E_1}\right) = \frac{30}{75} = \frac{2}{5}$

\therefore The probability that the first marble is red

and second marble is white.

$$= P(E_1 \cap E_2) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right)$$

$$= \frac{2}{15} \cdot \frac{2}{5} = \frac{4}{75}$$

(10)

⑥ The probabilities that students A, B, C, D solve a problem are $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{5}$, and $\frac{1}{4}$ respectively. If all of them try to solve the problem. What is the probability that the problem is solved?

Given the probability of A, B, C, D solving the problem is $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{5}$, $P(C) = \frac{1}{5}$, $P(D) = \frac{1}{4}$

The probability that the problem is not solved by ABCD are $P(\bar{A}) = \frac{2}{3}$, $P(\bar{B}) = \frac{3}{5}$, $P(\bar{C}) = \frac{4}{5}$ and $P(\bar{D}) = \frac{3}{4}$

The probability that the problem is not solved when A, B, C, D try together (independently) = $P(\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D})$

$$= P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C}) \cdot P(\bar{D}).$$

$$= \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{4}{5} \cdot \frac{3}{4} = \frac{6}{25}$$

\therefore The probability that the problem is solved.

$$= 1 - \frac{6}{25} = \frac{19}{25}.$$

⑦ A, B, C are aiming to shoot a balloon. A will succeed 4 times out of 5 attempts. The chance of B to shoot the balloon is 3 out of 4 and that of C is 2 out of 3. If the three aim the balloon simultaneously. Then find the probability that at least two of them hit the balloon.

Soln:- The probability of A hitting the target = $P(A) = \frac{4}{5}$

The probability of B hitting the target = $P(B) = \frac{3}{4}$

The probability of C hitting the target = $P(C) = \frac{2}{3}$

\therefore The probability of A, B, C not hitting the target respectively are $P(\bar{A}) = \frac{1}{5}$, $P(\bar{B}) = \frac{1}{4}$, $P(\bar{C}) = \frac{1}{3}$

Now the probability that exactly two will hit the balloon

$$= P(A \cap B \cap \bar{C}) + P(A \cap B \cap C) + P(\bar{A} \cap B \cap C)$$

$$= P(A) \cdot P(B) \cdot P(\bar{C}) + P(A) \cdot P(\bar{B}) \cdot P(C) + P(\bar{A}) \cdot P(B) \cdot P(C).$$

$$\text{outcomes} = \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} + \frac{4}{5} \cdot \frac{1}{4} \cdot \frac{2}{3} + \frac{1}{5} \cdot \frac{3}{4} \cdot \frac{2}{3}$$

$$= \frac{1}{5} + \frac{2}{15} + \frac{1}{10}$$

$$= \frac{6+4+3}{30} = \frac{13}{30}$$

The probability that all will hit the balloon

$$= P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

$$= \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{2}{5}$$

∴ The probability that at least two of them will hit the target

$$= \frac{13}{30} + \frac{2}{5}$$

$$= \frac{13+12}{30} = \frac{25}{30} = \frac{5}{6}$$

Q8 Two dice are thrown, let A be the event that the sum of the points on the faces is 9. Let B be the event that at least one number is 6. Find

- (i). $P(A \cap B)$ (ii). $P(A \cup B)$, (iii). $P(A^c \cup B^c)$.

SOLN :~ There are 36 simple outcomes when two dice are thrown

The event A that a sum 9 occurs in the

following way $A = \{(3,6), (4,5), (5,4), (6,3)\}$

$$\therefore P(A) = \frac{4}{36}$$

The event B that at least one number is 6

occurs in the following way (P. 360 Pg - 20/69)

$$B = \{(6,1) (6,2) (6,3) (6,4) (6,5) (6,6), (1,6) (2,6) (3,6) (4,6) (5,6)\}$$

$$P(B) = \frac{11}{36}$$

$$\text{Now } A \cap B = \{(3,6) (6,3)\}$$

$$(i) P(A \cap B) = \frac{2}{36} = \frac{1}{18}$$

$$(ii) P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{4}{36} + \frac{11}{36} - \frac{2}{36} = \frac{13}{36} = \frac{1}{2}$$

$$(iii) P(A^c \cup B^c) = P(A \cap B)^c = 1 - P(A \cap B) = \frac{17}{18}$$

$$= 1 - \frac{1}{18} = \frac{17}{18}$$

(9) In a group consisting of equal number men and women 10% of the men and 45% of the women are unemployed. If a person is selected randomly from the group then find the probability that the person is an employee.

Soln :- We have $P(m) = \frac{1}{2}$, $P(w) = \frac{1}{2}$

Let A be the event of employed and A^c be the unemployed event then

$$P(A^c_m) = \frac{10}{100} = \frac{1}{10}, P(\frac{e}{m}) = \frac{9}{10}$$

$$P(A^c_w) = \frac{45}{100} = \frac{9}{20}, P(\frac{e}{w}) = \frac{5.5}{10}$$

Probability that the person is employed.

$$= P(m) \cdot P(\frac{e}{m}) + P(w) \cdot P(\frac{e}{w}) \Rightarrow \frac{1}{2} \cdot \frac{9}{10} + \frac{1}{2} \cdot \frac{5.5}{10}$$

$$= \frac{1}{2} \left[\frac{9}{10} + \frac{5.5}{10} \right] = \frac{1}{2} \left[\frac{14.5}{10} \right] = \frac{14.5}{20}$$

$$= 0.725.$$

(10) The probability that India wins a cricket test match against West Indies is known to be $\frac{2}{5}$. If India and West Indies play three test matches, what is the probability that-

- India will lose all the three matches
- India will win at least one test match
- India will win at most one match.

Soln:- Given that probability of India winning against West Indies = $P(W) = \frac{2}{5}$

Probability of losing against West Indies = $P(\bar{W}) = \frac{3}{5}$

(i) The probability that India will loose all the three matches = $P(\bar{W}) \cdot P(\bar{W}) \cdot P(\bar{W})$
 $= \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$

(ii) The probability that India will win at least two matches = $1 - \frac{27}{125} = \frac{98}{125}$

(iii) The probability that India will win at most one match = The probability that India will win one match
 (or) no matches

$$= P(W) \cdot P(\bar{W}) \cdot P(\bar{W}) + P(\bar{W}) \cdot P(W) \cdot P(\bar{W})$$

$$= \frac{2}{5} \cdot \frac{3}{5} \cdot \frac{3}{5} + \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{3}{5}$$

$$= \frac{45}{125} = \frac{9}{25}$$

$$= \frac{9}{25} //$$

(ii) Three machines I, II, III produce 40%, 30%, 30% of the total number of items of factory. The percentages of defective items of these machines are 4%, 2%, 3%. If an item is selected at random. Find the probability that the item is defective.

Let A, B and C be the events that the machines I, II & III be chosen respectively and let D be the event which denotes the defective item.

$$\text{Given that } P(A) = \frac{40}{100} = 0.40, P(B) = \frac{30}{100} = 0.30 \\ \text{and } P(C) = \frac{30}{100} = 0.30 \text{ now find } P(D)$$

From the given data we have, $P(D|A) = 0.04$

$$P(D|B) = \frac{2}{100} = 0.02, P(D|C) = \frac{3}{100} = 0.03$$

The probability that the selected item at random is defective is $= P(D) = P(A) \cdot P(D|A) + P(B) \cdot P(D|B) + P(C) \cdot P(D|C)$

$$\text{Now } P(D) = \frac{40}{100} \times \frac{4}{100} + \frac{30}{100} \times \frac{2}{100} + \frac{30}{100} \times \frac{3}{100} \\ = \frac{16}{1000} + \frac{6}{1000} + \frac{9}{1000}$$

$$= \frac{16+6+9}{1000}$$

$$\therefore P(D) = \frac{31}{1000} \text{ or } 3.1\%$$

$$\text{The answer is } \frac{31}{1000} \text{ or } 3.1\% \text{ to be noted that it is not}$$

the required probability for an item to be defective. It is because the probability of an item being defective depends on the machine that produced it and not on the machine that selected it.

MOMENTS AND MOMENT GENERATING FUNCTIONS

Moments ~ Moments is defined as the arithmetic mean of the various powers of the derivations of items from their mean (assumed or actual). It will give the required power of moment of the distributions. If the deviations of the items are taken from the arithmetic mean of the distribution it is known as "Central moment". If we take the mean of the first power of deviations we get the first moment about the mean and is denoted by U_1 . Mean of the second power of the deviations gives us the second moment about the mean and is denoted by U_2 . Similarly the third moment about the mean denoted by U_3 is the mean of the cubes of the deviations from the mean, thus mean of the r^{th} power of deviations gives us the r^{th} moment about mean. It is denoted by U_r .

CENTRAL MOMENTS (or) MOMENTS ABOUT ACTUAL mean

① Central moments for individual series ~

Let \bar{x} be the mean of the individual series

Let x be the deviation of x from its mean \bar{x} .

$$\text{i.e.: } x = x - \bar{x}.$$

Let N be the total numbers of items (or) observations of the given series. Then we have.

$$U_1 = \frac{\sum(x - \bar{x})}{N} = \frac{\sum x}{N}$$

$$U_2 = \frac{\sum(x - \bar{x})^2}{N} = \frac{\sum x^2}{N}$$

$$\text{Similarly } \mu_1 = \frac{\sum (x - \bar{x})^2}{N} = \frac{\sum x^2}{N}$$

① Central Moments for Frequency Distribution:

If n observations x_1, x_2, \dots, x_n occurring with frequency f_1, f_2, \dots, f_n respectively, then the arithmetic mean of the freq. distribution is given by \bar{x} where

$$\bar{x} = \frac{\sum f_i x_i}{N} \quad \text{and} \quad N = \sum f_i$$

Let $x_i = x_i - \bar{x}$ be the Deviation of x from its mean

$$\text{then } \mu_1 = \frac{\sum f_i (x_i - \bar{x})}{N} = \frac{\sum f_i x_i}{N}$$

similarly we have $\mu_2 = \frac{\sum f_i (x_i - \bar{x})^2}{N}$

$$\mu_2 = \frac{\sum f_i (x_i - \bar{x})^2}{N} = \frac{\sum f_i x_i^2}{N}$$

similarly $\mu_3 = \frac{\sum f_i (x_i - \bar{x})^3}{N}$

$$\mu_3 = \frac{\sum f_i (x_i - \bar{x})^3}{N} = \frac{\sum f_i x_i^3}{N}$$

Properties of Central Moments:

- * The First Moment about mean is always zero i.e., $\mu_1 = 0$.

- * The Second Moment about mean measures variance i.e., $\mu_2 = \sigma^2$ (or) standard deviation, $\sigma = \sqrt{\mu_2}$

- * The Third Moment about the mean measure skewness of a given distribution.

- (i). If $\mu_3 > 0$, the distribution is positively skewed

- (ii). If $\mu_3 < 0$ the distribution is negatively skewed

- (iii). If $\mu_3 = 0$, the distribution is symmetrical.

- * The Fourth moment about mean measures kurtosis of a frequency distribution.

- * Skewness and kurtosis of a distribution are calculated from μ_2, μ_3 and μ_4 and are given by.

$$\text{skewness} = \beta_1 = \frac{\mu_3^2}{\mu_2^3} \quad \text{and} \quad \text{kurtosis} = \beta_2 = \frac{\mu_4}{\mu_2^2} \quad \text{and}$$

From ① we have, $\sqrt{\beta_1} = \pm \frac{\mu_3}{\mu_2^{3/2}}$
 so, the sign of skewness depends upon the sign of μ_3 .

$$\begin{aligned}\mu_2 &= \beta_2 \cdot \mu_2^3 \\ \mu_3 &= \sqrt{\beta_1} \cdot \sqrt{\mu_2^3} \\ \mu_3 &= \sqrt{\beta_1} = \sqrt[3]{\beta_1} \\ &= \frac{\mu_3}{(\mu_2)^{3/2}} \\ &= \frac{\mu_3}{\mu_2^{3/2}}\end{aligned}$$

Raw Moments (or) Moments About An arbitrary origin ~

When the actual mean of a distribution is a fraction, then it is difficult to calculate moments by using the above formulae.

In such a case, we first compute moments about arbitrary origin 'A' and then convert these moments into the moments about the actual mean. Moments about the arbitrary origin is called "RAW Moments" and are denoted by the symbol ' μ_r '. Thus ' μ_r ' stands for first moments about arbitrary origin A and so on.

The r^{th} moment about the point A is given by $\mu_r = \frac{1}{N} \sum f_i (x_i - A)^r$

where $\mu_r = \frac{1}{N} \sum f_i d_i^r$ where $d_i = x_i - A$

In the case of class interval frequency distribution, $\mu_r = \frac{1}{N} \sum f_i (d_i)^r$

where $d_i = \frac{x_i - A}{c}$, where $c = \text{length of interval}$

Relation b/w Moments about Mean in terms of Moments about any Point and vice-versa

(14)

$$\text{I. } \mu_1' = \mu_1 - \mu_1^2 = 0$$

$$\mu_2 = \mu_2' + (\mu_1')^2$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1 + 2(\mu_1')^3$$

$$\mu_4 = \mu_4' - \mu_2'\mu_1 + 6(\mu_1')^2\mu_2 - 3(\mu_1')^4$$

$$\text{II. } \mu_1' = \bar{x} - A$$

$$\mu_2' = \mu_2 + (\mu_1')^2$$

$$\mu_3' = \mu_3 + 3\mu_2\mu_1 + (\mu_1')^3$$

$$\mu_4' = \mu_4 + 4\mu_3\mu_1 + 6\mu_2(\mu_1')^2 + (\mu_1')^4$$

Purpose of Moments:-

From the above discussion, we conclude that the moments are very useful tools and techniques in statistical analysis. The measure of the characteristics of a frequency distribution such as mean, median, mode, standard deviation, skewness and kurtosis are directly connected with the first four central moments of a distribution. They can be obtained as given below

$$(i). \text{ Mean} = A + \mu_1'$$

$$(ii). \text{ Variance} = \mu_2$$

$$(iii). \text{ Skewness} = \beta_1 = \frac{\mu_3}{\mu_2^{3/2}}$$

co-efficient of skewness: $sk = \pm \frac{\mu_3}{\mu_2^{3/2}}$ Where the

sign of skewness sk is determined from the

sign of μ_3 .

$$(iv). \text{ Kurtosis} = \beta_2 = \frac{\mu_4}{\mu_2^2}$$

Moment Generating Function :-

It is a tool used to calculate the higher moments. The moment generating function of a random variable X , about the origin, whose prob density function $f_X(x)$ is given by

$M_X(t) = \begin{cases} \sum_{x=-\infty}^{\infty} e^{tx} p(x) & \text{in case of discrete random variable.} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x) dx & \text{in case of continuous random variable.} \end{cases}$

Since $M_X(t)$ is used to generate moments, it is known as moment generating function.

Now we shall discuss how $M_X(t)$ is used to generate the moments.

$$\text{We have } e^{tx} = 1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots$$

$$\therefore M_X(t) = E(e^{tx}) = E\left(1 + tx + \frac{t^2 x^2}{2!} + \frac{t^3 x^3}{3!} + \dots\right)$$

$$= 1 + tE(x) + \frac{t^2}{2!} E(x^2) + \frac{t^3}{3!} E(x^3) + \dots + \frac{t^r}{r!} E(x^r)$$

$$\text{So, required form} = 1 + \mu_1 t + \frac{t^2}{2!} \mu_2 t^2 + \dots + \frac{t^r}{r!} \mu_r t^r, \text{ where}$$

Where μ_r is the r th moment of x about the origin

$E(x^r) = \mu_r$ is the co-efficient of $\frac{t^r}{r!}$ in the expansion of $M_X(t)$.

In Order to find μ_r , we have to differentiate the moment 'r' times generating function $m_X(t)$.

'r' times with respect to 't' and substitute $t=0$

$$\text{i.e. } \mu_r = \left[\frac{d^r}{dt^r} [M_X(t)] \right]_{t=0}$$

In general the moment generating function of a random variable x about the point $x=a$ is define as

$$M_x(t) \text{ (about } x=a) = E[e^{t(x-a)}]$$

$$(e^{t(x-a)}) = E[1 + t(x-a) + \frac{t^2}{2!}(x-a)^2 + \dots + \frac{t^r}{r!}(x-a)^r]$$

$$= 1 + tM'_1 + \frac{t^2}{2!}M'_2 + \dots + \frac{t^r}{r!}M'_r + \dots$$

Where $M'_r = E\{(x-a)^r\}$ is the r^{th} moment about the point $x=a$. Then $f_x(x)$ is the density function of a continuous random variable. Then the moment generating function of this continuous probability distribution about $x=a$ is given by,

$$M_x(t) \text{ (about } x=a) = \int_{-\infty}^{\infty} e^{t(x-a)} f(x) dx.$$

Properties of Moment Generating Function :-

i. Let $y = ax+b$ where x is a random variable with moment Generating function $M_x(t)$.

$$\text{then } m_y(y) = m_x(ax+b) = e^{bt} m_x(at)$$

Proof :- By definition, we have

$$m_y(y) = M_x(ax+b) = E(e^{(ax+b)t}) = E(e^{axt} \cdot e^{bt}) = e^{bt} E(e^{axt})$$

$$= e^{bt} m_x(at).$$

2. $M_{kx}(t) = m_x(kt)$, where k is a constant.

Proof :- By definition, we have

$$m_{kx}(t) = E(e^{tkx}) = E(e^{x \cdot kt})$$

$$= m_x(kt).$$

3. If x and y are two independent random variables having the moment generating function $m_x(t)$ and $m_y(t)$ then the moment generating function of $(x+y)$ is given by $m_{x+y}(t) = m_x(t)m_y(t)$

Proof :- By definition, $M_{x+y}(t) = E(e^{t(x+y)})$ (1)

$$\text{Now, } M_{x+y}(t) = E(e^{t(x+y)}) = E(e^{tx+ty}) = E(e^{tx} \cdot e^{ty})$$
$$= E(e^{tx}) E(e^{ty})$$

But $E(e^{tx}) = M_x(t)$, $E(e^{ty}) = M_y(t)$, which

is to say that $M_{x+y}(t)$ is just the product of the moment generating function of the sum of two independent random variables.

Product of their respective moment generating functions.

The above result can be extended to a number of independent random variables.

4. A random variable X may have no moments even if its moment generating function exists.

5. A random variable X can have all its moments But moment generating function does not exist perhaps at one point.

Example :-

function of a random variable X is $E(e^{tX}) = e^{t^2}$

exists, condition for it to be

$$(e^{t^2}) < \infty \Rightarrow t^2 < \infty \Rightarrow t = 0$$

$$t = 0 \Rightarrow e^{t^2} = 1$$

which makes a function of X to be e^{t^2} .

which is not differentiable at $t = 0$ and hence

it is not a probability function and hence

from $(0, \infty) \rightarrow (0, \infty)$ is not

Probability Distribution $\approx \frac{1}{(0.5)^4} = 0.09$

(16)

In this chapter we will discuss binomial poisson
discrete random variable distributions to mostly other
and Normal distributions.

The 1st two models are for discrete random
variable and the last one is for continuous random
variable to understand this concept we need brief
introduction of Uniform and Bernoulli's distribution.

Discrete Uniform Distribution:-

A random variable 'x' has a discrete uniform
distribution if and only if its prob'y distribution is
given by $P(x) = \frac{1}{K}$ for $x=x_1, x_2, \dots, x_K$. Then 'x' is called
discrete uniform random variable.

Ex:- ①

x	0	1	2
P(x)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Example :- If x is a discrete random variable A

x	0	1	2	3
P(x)	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Bernoulli's Distribution:-

A random variable 'x' which takes two values 0,1 with prob's q & p respectively i.e $P(x=0)=q$ and $P(x=1)=p$ is called Bernoulli's discrete random variable & is said to be have a Bernoulli's distribution

The prob function is of Bernoulli's distribution

can be written as

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$$P(X) = p^x q^{1-x} \quad \text{for } x=0, 1.$$

Note :- Mean of Bernoulli's discrete random variable

is $\mu = np$. This is called the mean of Bernoulli distribution.

Similarly, Variance of Bernoulli distribution is given by $\sigma^2 = npq$.

$$(i). \text{ Mean } \mu = E(X) = \sum x_i p(x_i) = 0 \times q + 1 \times p = p.$$

$$(ii). \text{ Variance } (\sigma^2) = \sum x_i^2 p(x_i) - \mu^2 \\ = (0^2)(q) + (1^2)p - p^2 \\ = p - p^2 = p(1-p) = pq.$$

Standard deviation of Bernoulli distribution is

$$\sigma^2 = pq.$$

$$(iii). \text{ S.D } (\sigma) = \sqrt{pq}.$$

Binomial Distribution :-

A random variable $'X'$ has a binomial distribution if it assumes only non-negative values and its probability density function is given by

$$P(X=r) = P(r) = \begin{cases} {}^n C_r \cdot p^r \cdot q^{n-r} & r=0, 1, 2, \dots, n; \\ 0 & \text{Otherwise.} \end{cases}$$

Range of values taken by binomial random variable is $[0, n]$.
Mean of binomial random variable is np .
Variance of binomial random variable is npq .
Standard deviation of binomial random variable is \sqrt{npq} .

* Constants of Binomial distribution :-

Mean of Binomial distribution :-

The Binomial distribution is given by

$$P(r) = {}^n C_r \cdot p^r \cdot q^{n-r}, \quad r=0, 1, 2, \dots, n$$

$$\text{mean } (\mu) = E(X) = \sum_{r=0}^n r P(r)$$

$$= \sum_{r=0}^n r \cdot {}^n C_r \cdot p^r \cdot q^{n-r}$$

$$= 0 + 1 \cdot {}^n C_1 \cdot p^1 \cdot q^{n-1} + 2 \cdot {}^n C_2 \cdot p^2 \cdot q^{n-2} + \dots + n \cdot {}^n C_n \cdot p^n \cdot q^{n-n}$$

$$= n \cdot p \cdot q^{n-1} + 2 \frac{n(n-1)}{2!} p^2 \cdot q^{n-2} + \dots + n \cdot 1 \cdot p^n \cdot 1 \quad [:: 2^0 = 1]$$

$$= n \cdot p (q^{n-1} + (n-1)p \cdot q^{n-2} + \dots + p^{n-1})$$

$$= n \cdot p (p+q)^{n-1}$$

$$= n \cdot p \cdot (1)^{n-1}$$

$$\mu = n \cdot p$$

(ii). Variance of the Binomial Distribution :-

$$V(X) = E(X^2) - [E(X)]^2$$

$$\text{Total variation} = E(X^2) - \mu^2$$

$$\text{Expected value} = \sum_{r=0}^n r^2 P(r) = \mu^2$$

$$= \sum_{r=0}^n (r(r-1) + r) P(r) - \mu^2$$

$$= \sum_{r=0}^n r(r-1) P(r) + \sum_{r=0}^n r P(r) - \mu^2$$

$$= \sum_{r=0}^n r(r-1) {}^n C_r \cdot p^r \cdot q^{n-r} + E(X) - \mu^2$$

$$= 0 + 2(0) {}^n C_2 \cdot p^2 \cdot q^{n-2} + 3(2) {}^n C_3 \cdot p^3 \cdot q^{n-3} + \dots + n(n-1) {}^n C_n \cdot p^n \cdot q^{n-n} + \mu - \mu^2$$

$$= \frac{2n(n-1)}{2} p^2 q^{n-2} + \frac{3 \cdot 2 \cdot \dots \cdot n(n-1)(n-2)}{3 \times 2 \times 1} p^3 \cdot q^{n-3} + \dots - u - u^2$$

$$= n(n-1) p^2 [q^{n-2} + (n-2)p \cdot q^{n-3} + \dots + p^{n-2}] + u - u^2$$

3. Variance

[By Binomial theorem]

$$= n(n-1) p^2 [(q+p)^{n-2}] + u - u^2$$

$$= n(n-1) p^2 (1)^{n-2} + u - u^2$$

$$= n(n-1) p^2 + np - n^2 p^2$$

$$= (n^2 - n) p^2 + np - n^2 p^2$$

$$= np - np^2$$

$$= np(1-p)$$

$$\text{variance} = npq$$

Hence the standard deviation of the Binomial distribution is $\sigma = \sqrt{npq}$.

(iii). Mode of the Binomial distribution

Mode of the binomial distribution is the

value of x at which $P(x)$ has maximum value.

Mode = $\begin{cases} \text{integer part of } (n+1)p & \text{if } (n+1)p \text{ is not integer} \\ (n+1)p \text{ and } (n+1)p-1 & \text{if } (n+1)p \text{ is an integer.} \end{cases}$

Recurrence relation for Binomial distribution:

$$P(r) = {}^n C_r p^r q^{n-r} \quad \text{--- (1)}$$

$$P(r+1) = {}^n C_{r+1} p^{r+1} q^{n-r-1} \quad \text{--- (2)}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{P(r+1)}{P(r)} = \frac{{}^n C_{r+1}}{{}^n C_r} = \frac{p^{r+1}}{p^r} \cdot \frac{q^{n-r-1}}{q^{n-r}}$$

$$\therefore \frac{P(r+1)}{P(r)} = \frac{p}{q} = \frac{n-r}{r+1} \cdot \frac{p}{q}$$

$$\therefore P(r+1) = \frac{n-r}{r+1} \cdot \frac{p}{q} P(r).$$

Moments about Origin of the Binomial Distribution:-

The first four moments about origin of Binomial distribution are obtained as follows.

$$M_1' = E(X) = \sum_{x=0}^n x \cdot P(x).$$

$$= \sum_{x=0}^n x \cdot \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^n x \binom{n}{x} \frac{(n-1)}{(x-1)} p^x q^{n-x}$$

$$= \sum_{x=1}^n n \binom{n-1}{x-1} p^x q^{n-x} =$$

$$= \sum_{x=1}^{\infty} n \binom{n-1}{x-1} p^x \frac{p}{p} \cdot q^{n-x}.$$

$$= np \sum_{x=1}^{\infty} \binom{n-1}{x-1} p^x p^{-1} q^{n-x}.$$

$$= np \sum_{x=1}^{\infty} \binom{n-1}{x-1} p^{x-1} q^{n-x}.$$

$$= np (q+p)^{n-1}$$

$$= np. \quad [\because P(x) = \sum_{x=0}^n \binom{n}{x} p^x q^{n-x} = (q+p)^n]$$

Thus the mean of the Binomial distribution is np .

$$M_2' = E(X^2) = \sum_{x=0}^{\infty} x^2 P(x) = \sum_{x=0}^{\infty} x^2 \binom{n}{x} p^x q^{n-x}.$$

$$= \sum_{x=0}^{\infty} \{x(x-1)+x\} \frac{n}{x} \frac{n-1}{x-1} \binom{n-2}{x-2} p^x q^{n-x}.$$

$$= \sum_{x=0}^n x(x-1) \frac{n}{x} \cdot \frac{n-1}{x-1} \binom{n-2}{x-2} p^x q^{n-x} + \sum_{x=0}^n x \frac{n(n-1)}{x(n-1)} \binom{n-2}{x-2} p^x q^{n-x}$$

$$= n(n-1)p \left[\sum_{x=2}^n \binom{n-2}{x-2} p^{x-2} q^{n-x} \right] + np \quad (\text{from } M_1')$$

$$= n(n-1)p^2 \left[\sum_{x=2}^n \binom{n-2}{x-2} p^{x-2} q^{n-x} \right] + np.$$

$$\therefore \text{unit-4-Pg-35/69} \quad = n(n-1)p^2 (q+p)^{n-2} + np = n(n-1)p^2 + np.$$

$$\mu_3 = E(x^3) = \sum p_i x_i^3$$

to write $\sum p_i x_i^3$ in terms of p & q

$$= \sum_{x=0}^n x^3 \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^n [x(x-1)(x-2) + 3x(x-1) + x] \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^n x(x-1)(x-2) \binom{n}{x} p^x q^{n-x} + 3 \sum_{x=0}^n x(x-1) \binom{n}{x} p^x q^{n-x}$$

$$+ \sum_{x=0}^n x \binom{n}{x} p^x q^{n-x}$$

$$= n(n-1)(n-2) p^3 \sum_{x=3}^n \binom{n-3}{x-3} p^{x-3} q^{n-x} + 3n(n-1)p^2 \sum_{x=2}^n \binom{n-2}{x-2} p^{x-2} q^{n-x}$$

$$= n(n-1)(n-2) p^3 (q+p)^{n-3} + 3n(n-1)p^2 (q+p)^{n-2} + np.$$

$$= n(n-1)(n-2) p^3 + 3n(n-1)p^2 + np.$$

($\because p+q=1$)

similarly

$$\mu_4 = E(x^4) = \sum_{x=0}^n x^4 \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^n [x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x^2]$$

$$= \sum_{x=0}^n x(x-1)(x-2)(x-3) \binom{n}{x} p^x q^{n-x}$$

$$= n(n-1)(n-2)(n-3) p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np.$$

Central Moments of Binomial Distribution

$$\mu_2 = \mu_2' - (\mu_1')^2 = [n(n-1)p^2 + np] - (np)^2$$

$$= (n^2-n)p^2 + np - n^2p^2$$

$$= \frac{1}{4}p^2 - np^2 + np - n^2p^2$$

$$= np - np^2$$

$$= np(1-p)$$

$$= np(2).$$

$$\mu_2 = npq \quad \text{without derivation} \quad [., q+p=1]$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3.$$

$$= n[(n-1)(n-2)p^3 + 3n(n-1)p^2 + np] - 3[n(n-1)p^2 + np][np].$$

$$= np[-3np^2 + 3np + 2p^2 - 3p + 1 - 3np^2] + 2(np)^3$$

$$= np[3np(1-p) + 2p^2 - 3p + 1 - 3np^2]$$

$$= np[3npq + 2p^2 - 3p + 1 - 3np^2]$$

$$= np[3npq + 2p^2 - 3p + 1 - 3np^2]$$

$$= np[2p^2 - 3p + 1]$$

$$= np[2p^2 - 2p + (1-p)]$$

$$= np[2p^2 - 2p + 2] = np(-2p(1-p) + 2)$$

$$= np[-2pq + 2] = npq(1-2p)$$

$$= npq(2+p-2p) = npq(2-p)$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= npq(1+3(n-2)pq)$$

Moment Generating Function of a Binomial Distribution

If x is a binomial distribution variable then the moment generating function is $(pe^t + q)^n$.

Proof :- We know that

The Binomial density function of the Binomial

Distribution is given by

$$P\{x=x\} = P(x) = \begin{cases} nC_x \cdot p^x \cdot q^{n-x} & x=0, 1, 2, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

The moment Generating function is

$$m_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^n e^{tx} n C_x p^x q^{n-x}$$

$$= \sum_{x=0}^n n C_x (pe^t)^x q^{n-x}$$

In accordance with Binomial theorem,

$$[q^n + q^{n-1} + \dots + q + 1] [q^n + q^{n-1} + \dots + q + 1]$$

$$\text{we have } m_x(t) = (pe^t + q)^n.$$

Moment Generating Function about mean of Binomial Distribution:

$$+ [1 + q - pe^t] q^n$$

We know that the moment Generating

function of a random variable x about the point

$$x=a \text{ is } m_x(t) = E(e^{t(x-a)})$$

\therefore Moment generating Function about mean of Binomial distribution.

$$\begin{aligned} &E(e^{t(x-a)}) = E(e^{t(x-np)}) \\ &= E(e^{tx - tnp}) \end{aligned}$$

$$= e^{-tnp} \cdot E(e^{tx}) = e^{-tnp} m_x(t)$$

With addition of independent term t

$$= e^{-tnp} (q + pe^t)^n \quad [\because m_x(t) = (q + pe^t)^n]$$

$$= [q e^{-tnp} + pe^t (1-p)^n]$$

$$= [q e^{-tnp} + pe^t q^n]^n.$$

Comparing with $(q + pe^t)^n$ binomial distribution

$$= \left[q \left\{ 1 - pt + \frac{p^2 t^2}{2!} - \frac{p^3 t^3}{3!} + \frac{p^4 t^4}{4!} - \dots \right\} + p \left\{ 1 + tq + \frac{t^2 q^2}{2!} - \dots \right\} \right]^n$$

$$= \left[q \left\{ 1 - pt + \frac{p^2 t^2}{2!} - \frac{p^3 t^3}{3!} + \frac{p^4 t^4}{4!} - \dots \right\} + p \left\{ 1 + tq + \frac{t^2 q^2}{2!} + \frac{t^3 q^3}{3!} + \dots \right\} \right]^n$$

$$\therefore \text{Using } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$[\because \text{Using } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots]$$

$$(q+p)^n = \left[1 + \frac{t^2}{2!} p^2 + \frac{t^3}{3!} (q^2 - p^2) pq + \frac{t^4}{4!} pq (q^3 + p^3) + \dots \right]^n$$

$$= \left[1 + \sum \left\{ \frac{t^2}{2!} pq + \frac{t^3}{3!} pq (q-p) + \frac{t^4}{4!} (1-3p) + \dots \right\} \right]^n$$

$$= 1 + n c_1 \left\{ \frac{t^2}{2!} pq + \frac{t^3}{3!} pq (q-p) \right\} + n c_2 \left\{ \frac{t^2}{2!} pq + \frac{t^3}{3!} pq (q-p) \right\}^2 + \dots$$

$$= 1 + npq \cdot \frac{t^2}{2!} + npq (q-p) \frac{t^3}{3!} + 3n^2 p^2 q^2 + npq (1-3p) \frac{t^4}{4!} + \dots$$

compares from $1 + t \mu_1 + \frac{t^2}{2!} \mu_2 + \dots + \frac{t^r}{r!} \mu_r + \dots$.

$$\text{Now } \mu_2 = \text{co-efficient of } \frac{t^2}{2!} = npq$$

$$\mu_3 = \text{co-efficient of } \frac{t^3}{3!} = npq (q-p)$$

$$\mu_4 = \text{co-efficient of } \frac{t^4}{4!} = npq (1-3p) + 3n(n-1)p^2 q^2$$

$$= npq (1-3p) + 3n^2 p^2 q^2 - 3np^2 q^2$$

$$= 3n^2 p^2 q^2 + npq (1-pq).$$

Binomial Frequency distribution:

In 'n' independent trials consistute one experiment and this experiment is repeated N times. Then the frequency of r successes is $N \cdot n c_r \cdot p^r \cdot q^{n-r}$. since the probability of 0, 1, 2, 3, ..., r, ..., n successes in n trials are given by the terms of the binomial expansion of $(q+p)^n$. Therefore in N sets of n trials the terms of expansion of $N(q+p)^n$. The possible number of success and their frequencies is called a Binomial frequency Distribution.

① Ten coins are thrown simultaneously. Find the probability of getting at least (i). Seven heads (ii). Six heads.

$$\text{Soln: } p = \text{Prob of getting a head} = \frac{1}{2}$$

$$q = \text{Prob of not getting a head} = \frac{1}{2}$$

The prob of getting x heads in a throw of 10 coins is $P(x=r) = P(r) = {}^{10}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{10-r}$ for $r=0, 1, 2, 10$

(i) Prob of getting at least seven heads is given by $P(x \geq 7)$

$$= P(x=7) + P(x=8) + P(x=9) + P(x=10)$$

$$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^{10-7} + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{10-8} + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^{10-9}$$

$$+ {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^{10-10}$$

$$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^1 + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$$

$$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^1 + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$$

$$= \frac{1}{2^{10}} \left({}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \right)$$

$$= \frac{1}{2^{10}} (120 + 45 + 10 + 1)$$

$$= \frac{176}{1024} = 0.1719$$

(ii) This is left as an exercise

$$= P(x=6) + P(x=7) + P(x=8) + P(x=9) + P(x=10)$$

similarly

$$= \frac{1}{2^{10}} \left[{}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \right]$$

(2) Out of 800 families with 5 children each, how many would you expect to have (a) 3 boys (b) 5 girls

(c) either 2 or 3 boys.

(d) at least one boy? Assume equal probability for boys and girls.

Soln :- Let the no. of boys in each family = x .

P = The probability of each boy = $\frac{1}{2}$.

P = The probability distribution [\because since equal probability for boys and girls]
is $P(r) = {}^n C_r p^r q^{n-r}$

No. of children, $n=5$:

$$= {}^5 C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{5-r}$$

$$= \frac{1}{2^5} {}^5 C_r \text{ per family.}$$

(a). $P(3 \text{ boys}) = P(r=3)$

$$= P(3) = \frac{1}{2^5} {}^5 C_3$$

$$= \frac{10}{32} = \frac{5}{16} \text{ per family}$$

Thus for 800 family the Probability of number of families having 3 boys

$$= \frac{5}{16} (800) = 250 \text{ families.}$$

(b) $P(5 \text{ girls}) = P(\text{no boys}) = P(r=0)$

$$= P(r) = \frac{1}{2^5} {}^5 C_0 = \frac{1}{32} \text{ per family}$$

Thus for 800 families the probability of number of families having 5 girls.

$$= \frac{1}{32} (800) = 25 \text{ families.}$$

$$(c) P(\text{either 2 or 3 boys}) = P(r=2) + P(r=3)$$

$$\begin{aligned} &= \frac{1}{2^5} \left({}^5C_2 + {}^5C_3 \right) \\ &= \frac{1}{2^5} (10 + 10) \\ &= \frac{20}{32} = \frac{5}{8} \text{ per family.} \end{aligned}$$

∴ Expected no. of families with 2 or 3 boys

$$= \frac{5}{8} (800) = 500 \text{ families.}$$

$$(d) P(\text{at least one boy}) = P(r=1) + P(r=2) + P(r=3) + P(r=4) + P(r=5).$$

$$= \frac{1}{2^5} \left({}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 \right)$$

$$= \frac{1}{2^5} (5 + 10 + 10 + 5 + 1)$$

$$= \frac{31}{32}$$

∴ Expected no. of Families with atleast one boy

$$= \frac{31}{32} (800) = 775 \text{ families.}$$

Poisson Distribution ~ (Derivation) :

(2)

- Def ~ The Poisson distribution can be derived as a limiting case of binomial distribution under the conditions that (i). p is very small.
(ii). n is very large.
(iii). $np = \lambda$ (say) is finite.

In the Binomial distribution the probability $P(r)$ of r success is given by $P(r)$.

$$\begin{aligned}
P(r) &= {}^n C_r p^r q^{n-r} \\
&= \frac{n(n-1)(n-2)\dots(n-(r-1))}{r!} p^r (1-p)^{n-r} \\
&= \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} p^r \frac{(1-p)^r}{(1-p)^r} \\
\because n &= \frac{1}{p} \\
&= \frac{\frac{1}{p} \left(\frac{1}{p}-1\right) \left(\frac{1}{p}-2\right) \dots \left(\frac{1}{p}-r+1\right)}{r!} p^r \frac{(1-p)^r}{(1-p)^r} \\
&= \frac{\frac{1}{p} \left(\frac{1-p}{p}\right) \left(\frac{1-2p}{p}\right) \dots \left(\frac{1-pr+p}{p}\right)}{r!} p^r \frac{(1-p)^r}{(1-p)^r} \\
&= \frac{\lambda(\lambda-p)(\lambda-2p)\dots(\lambda-(r-1)p)}{r!} p^r \frac{(\frac{1-p}{1-p})^r}{(\frac{1-p}{1-p})^r}
\end{aligned}$$

as $n \rightarrow \infty$ $\lambda_n \rightarrow 0$ $p \rightarrow 0$

$$\begin{aligned}
&\text{If } n \rightarrow \infty \quad \left[\frac{\lambda(\lambda-p)(\lambda-2p)\dots(\lambda-(r-1)p)}{r!} \right] \frac{(1-p)^r}{(1-p)^r} \\
&= \frac{\lambda(\lambda-0)(\lambda-0)\dots(\lambda-0)}{r!} \underset{n \rightarrow \infty}{\text{If}} \left(1 - \frac{1}{n}\right)^n \underset{p \rightarrow 0}{\text{If}} \frac{1}{(1-p)^r} \\
&= \frac{\lambda^r}{r!} e^{-\lambda} \cdot 1 = \frac{\lambda^r e^{-\lambda}}{r!}
\end{aligned}$$

$$\therefore P(r) = \text{Probability of Success} = \frac{e^{-\lambda} \lambda^r}{r!}$$

This is known as Poisson distribution put $r=0, 1, 2, \dots$

The probability of $0, 1, 2, \dots$ success are given by

$e^{-\lambda}, e^{-\lambda} \cdot \frac{\lambda^1}{1!} e^{-\lambda}, \dots$ respectively, where $\lambda (>0)$ is a parameter.

Definition of a Poisson distribution :-

A random variable 'x' is said to be to follow a Poisson distribution if it assumes only non-negative values and its probability distribution is given by.

$$P(x=x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!} & : x=0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\lambda (>0)$ is a parameter of distribution.

Constants of poisson distribution :-

$$\text{Mean} = E(x) = \mu = \sum_{x=0}^{\infty} x \cdot P(x)$$

$$\begin{aligned} E(x) &= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^x}{x!} \\ &= \sum_{x=1}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^x}{x!} = \sum_{x=1}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^x}{x(x-1)!} \end{aligned}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!} = e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$\text{Put } y = x-1 \Rightarrow e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!}$$

$$= e^{-\lambda} \cdot 1 \cdot \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} \quad \left[\because \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = e^{\lambda} \right]$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda = np$$

$$\left[\therefore 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots = \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = e^{\lambda} \right]$$

Variances:

$$\begin{aligned}
 V(X) &= \sigma^2 = E(X^2) - [E(X)]^2 \\
 &= \sum_{x=0}^{\infty} x^2 p(x) - \mu^2 \\
 &= \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda} \lambda^x}{x!} - \mu^2 \\
 &= \sum_{x=1}^{\infty} \frac{x \cdot x \cdot e^{-\lambda} \lambda^x}{x(x-1)!} - \mu^2 \\
 &= \sum_{x=1}^{\infty} \frac{x e^{-\lambda} \lambda^x}{(x-1)!} - \lambda^2 \Rightarrow \sum_{x=1}^{\infty} \frac{(x-1+1) e^{-\lambda} \lambda^x}{(x-1)!} - \lambda^2 \\
 &= \sum_{x=1}^{\infty} \frac{(x-1) e^{-\lambda} \lambda^x}{(x-1)!} + \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!} - \lambda^2 \\
 &= \sum_{x=2}^{\infty} \frac{(x-1) e^{-\lambda} \lambda^x}{(x-1)(x-2)!} + \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!} - \lambda^2 \\
 &= e^{-\lambda} \left[\sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} + \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \right] - \lambda^2 \\
 \text{Put } y = x-2, z = x-1 & \\
 &= e^{-\lambda} \left[\sum_{y=0}^{\infty} \frac{\lambda^{y+2}}{y!} + \sum_{z=0}^{\infty} \frac{\lambda^{z+1}}{z!} \right] - \lambda^2 \\
 &= e^{-\lambda} \left[\lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} + \lambda \sum_{z=0}^{\infty} \frac{\lambda^z}{z!} \right] - \lambda^2 \\
 &= e^{-\lambda} [\lambda^2 e^\lambda + \lambda e^\lambda] - \lambda^2 \\
 &= \lambda^2 + \lambda - \lambda^2 \\
 V(X) &= \lambda.
 \end{aligned}$$

Mode of Poisson Distribution :-

Mode is the value of x for which the probability $P(x)$ is maximum.

$$\therefore P(x) \geq P(x+1) \text{ and } P(x) \geq P(x-1).$$

$$\text{Now } P(x) \geq P(x+1) \Rightarrow \frac{e^{-\lambda} \lambda^x}{x!} \geq \frac{e^{-\lambda} \lambda^{x+1}}{(x+1)!}$$

$$= \lambda \geq \frac{1}{x+1}$$

$$= \lambda \leq x+1 \quad (\text{or}) \quad x+1 \geq \lambda$$

$$\text{Similarly } P(x) \geq P(x-1)$$

$$x \leq \lambda - ②$$

Combining ① and ② we have

$$\lambda - 1 \leq x \leq \lambda$$

Hence mode of the Poisson distribution lies between $\lambda - 1$ and λ .

Recurrence Relation for the Poisson Distribution :-

$$\text{We have } P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$P(x+1) = \frac{e^{-\lambda} \lambda^{x+1}}{(x+1)!} = \frac{\lambda}{(x+1)} \cdot \frac{e^{-\lambda} \lambda^x}{x!} = \frac{\lambda}{(x+1)} P(x)$$

$$\therefore P(x+1) = \left(\frac{\lambda}{x+1}\right) P(x)$$

$$=$$

(24)

Properties of the Poisson Distribution :-

1. Range of the variable is from 0 to ∞ .
2. Mean and Variance are equal.
3. Distribution gets more and more symmetrical about the mean as d increases and tends to normal distribution.

Moments about Origin of the Poisson Distribution :-

We have $p(x, d) = \begin{cases} \frac{e^{-d} d^x}{x!} & : x = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$

$$\mu_1 = E(x) = \sum_{x=0}^{\infty} x p(x, d).$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-d} d^x}{x!} = d e^{-d} \left[\sum_{x=1}^{\infty} \frac{d^{x-1}}{(x-1)!} \right].$$

$$= d e^{-d} \left[1 + d + \frac{d^2}{2!} + \frac{d^3}{3!} + \dots \right]$$

$$= d e^{-d} e^d = d e^{-d+1}$$

Here the mean of the Poisson distribution is d .

$$\mu_2 = E(x^2) = \sum_{x=0}^{\infty} x^2 p(x, d)$$

$$= \sum_{x=0}^{\infty} [x(x-1) + x] \frac{e^{-d} d^x}{x!}$$

$$= e^{-d} \sum_{x=0}^{\infty} x(x-1) \frac{d^x}{x!} + \sum_{x=0}^{\infty} x \cdot \frac{e^{-d} d^x}{x!}$$

$$= d^2 e^{-d} \sum_{x=2}^{\infty} \frac{d^{x-2}}{(x-2)!} + d \quad (\because \text{From } \mu_1 = d)$$

$$= d^2 e^{-d} e^d + d$$

$$\mu_2' = d^2 + d.$$

$$M_3' = e(x^3) = \sum_{k=0}^{\infty} x^3 p(x,k)$$

$$= \sum_{k=0}^{\infty} [x(x-1)(x-2) + 3x(x-1) + x] \cdot \frac{e^{-d} d^x}{x!}$$

$$= \sum_{k=0}^{\infty} x(x-1)(x-2) \frac{e^{-d} d^x}{x!} + 3 \sum_{k=0}^{\infty} x(x-1) \frac{e^{-d} d^x}{x!} +$$

$$\sum_{k=0}^{\infty} x \cdot \frac{e^{-d} d^x}{x!}$$

$$= e^{-d} d^3 \left[\sum_{k=3}^{\infty} \frac{d^{x-3}}{(x-3)!} \right] + 3 e^{-d} d^2 \left[\sum_{k=2}^{\infty} \frac{d^{x-2}}{(x-2)!} \right] + 1$$

$$= e^{-d} (d^3 e^d + 3 e^{-d} d^2 e^d + 1)$$

$$M_3' = d^3 + 3d^2 + d.$$

$$M_4' = e(x^4) = \sum_{k=0}^{\infty} x^4 p(x,k).$$

$$= \sum_{k=0}^{\infty} [x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7(x-1) + x]$$

$$\frac{e^{-d} d^x}{x!}$$

$$= e^{-d} d^4 \left[\sum_{k=4}^{\infty} \frac{d^{x-4}}{(x-4)!} \right] + 6 e^{-d} d^3 \left[\sum_{k=3}^{\infty} \frac{d^{x-3}}{(x-3)!} \right] +$$

$$= e^{-d} (d^4 e^d + 6 e^{-d} d^3 e^d + 7 e^{-d} d^2 e^d + 1)$$

$$= d^4 + 6d^3 + 7d^2 + d.$$

$$\frac{d^4}{dx^4} e^{-d} x^4 = d^4 =$$

$$1 + \frac{d^2}{(x-2)} - \frac{d^3}{(x-3)} +$$

$$= d^4 + 6d^3 + 7d^2 + d$$

Central moments of poisson Distribution:

The four Central moments of poisson distribution are now obtained as follows :

$$\text{According to } M_1 = 0 \text{ (from given)} \quad \text{Also } M_0 = 1$$

$$M_2 = M_2^1 - (M_1)^2 = (M_1)^2 + 0^2 = 1^2 = 1.$$

$$M_3 = M_3^1 - 3M_1^1 M_2^1 + 2(M_1^1)^3$$

$$= (1^3 + 3) - 3(1)(1^2 + 1) + 2(1^3)$$

$$= 1^3 + 3 \cdot 1^2 + 1 - 3 \cdot 1^3 - 3 \cdot 1^2 + 2 \cdot 1^3$$

$$M_3 = 1$$

$$M_4 = M_4^1 = 6M_3^1 M_1^1 + 6M_2^1 (M_1^1)^2 - 3(M_1^1)^4$$

$$= (1^4 + 6 \cdot 1^3 + 7 \cdot 1^2 + 1) - 4 \cdot 1 (1^3 + 3 \cdot 1^2 + 1) + 6 \cdot 1^2 (1^2 + 1) - 3 \cdot 1^4$$

$$= 1^4 + 6 \cdot 1^3 + 7 \cdot 1^2 + 1 - 4 \cdot 1^4 - 12 \cdot 1^3 - 4 \cdot 1^2 + 6 \cdot 1^4 + 6 \cdot 1^3 - 3 \cdot 1^4$$

$$\text{So } M_4 = 3 \cdot 1^2 + 1^4 = 3 \cdot 1 + 1 = 4$$

Moment Generating Function of Poisson Distribution:

If x is a Poisson distribution variable, Then its moment generating function is $e^{t\lambda}(e^t - 1)$.

Proof:

Let x be the random variable having a poisson distribution with the probability density function given by.

$$P(x=x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!} & : (x=0, 1, 2, \dots) \\ 0 & \text{otherwise.} \end{cases}$$

additional notes to student India
The corresponding "poisson" distribution is given

by $F_x(x) = P(X \leq x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!}$

The moment Generating function of the poisson distribution is given by $m_x(t) = E(e^{tx})$

$$\begin{aligned} m_x(t) &= e^{-\lambda} \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k}{k!} \\ &= e^{-\lambda} + \sum_{k=1}^{\infty} e^{tk} \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(te^{\lambda})^k}{k!} \end{aligned}$$

$$m_x(t) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(te^{\lambda})^k}{k!}$$

$$m_x(t) = e^{-\lambda} \left[1 + \frac{te^{\lambda}}{1!} + \frac{(te^{\lambda})^2}{2!} + \dots \right]$$

$$= e^{-\lambda} e^{te^{\lambda}}$$

$$m_x(t) = e^{-\lambda} (e^t - 1)$$

① A car-hire firm has two cars which it hires out day by day. The no. of demands for a car on each day is distributed as a poisson distribution with mean 1.5 calculate the proportion of days.

(i), On which there is 'no' demand

(ii), On which 'demand' is refused.

Soln :- Given mean $\lambda = 1.5$

$$P(r) = \frac{e^{-\lambda} \lambda^r}{r!}$$

$$(i) P(\text{no demand}) = P(0) = \frac{e^{-1.5} (1.5)^0}{0!}$$

$$= e^{-1.5}$$

$$= 0.2231 \quad \text{unit-1 Pg-56/69}$$

Note :- No. of days in a year there is no demand
 \therefore No. of days in a year of 365 days in which
 of car = 365 (0.2231) = 81 days.

(ii). Some demand is refused if the no. of demands
 is more than two i.e. $r > 2$.

$$P(\text{demand refused}) = P(r \geq 3) = 1 - [P(0) + P(1) + P(2)].$$

$$= 1 - \left[e^{-1.5} + \frac{e^{-1.5}(1.5)}{1!} + \frac{e^{-1.5}(1.5)^2}{2!} \right].$$

$$= 1 - e^{-1.5} [1 + 1.5 + 1.25]$$

$$\therefore \text{Refused} = 1 - 3 \cdot 625 (e^{-1.5})$$

$$= 1 - 0.8088 = 0.1913.$$

Note :- No. of days in a year demand is refused

$$= 365 \times 0.1913 = 69.82$$

$$= 70 \text{ days.}$$

② A manufacturer of caterpins knows that 5% of his product is defective pins are sold in boxes of 100. He guarantees that not more than 10 pins will be defective. What is the approximate probability that a box will fail to meet guaranteed quantity?

Soln :- The probability of cater pins to be defective

$$is p = 5\% = 0.05.$$

Total number of cater pins $n = 100$

$$\therefore \text{mean } d = np = 100 (0.05) = 5$$

$$\text{we have } P(X=x) = \frac{e^{-d} d^x}{x!}$$

$$P(X=x) = \frac{e^{-5} 5^x}{x!}$$

$$= P(\text{box will fail to meet the guarantee})$$

$$= P(X > 10) = 1 - P(X \leq 10)$$

$$= 1 - [P(X=0) + P(X=1) + \dots + P(X=10)]$$

$$[(0.9 + 0.09 + 0.009) + \dots + \frac{e^{-5}(5)^0}{0!} + \frac{e^{-5}(5)^1}{1!} + \dots + \frac{e^{-5}(5)^{10}}{10!}]$$

$$= 1 - \left[\frac{e^{-5}(5)^0}{0!} + \frac{e^{-5}(5)^1}{1!} + \dots + \frac{e^{-5}(5)^{10}}{10!} \right]$$

$$= 1 - 0.9863$$

$$= 0.0137$$

③ Fit a poisson distribution to the following data.

x	0	1	2	3	4	5	Total
f	142	156	69	27	5	156	400

$$\text{Soln: Mean } \bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{0(142) + 1(156) + 2(69) + 3(27) + 4(5)}{400}$$

$$= \frac{156 + 138 + 81 + 20 + 5}{400}$$

Off. no. of boxes failing = 400

$$\text{Required no. of boxes failing} = \frac{400}{400} = 1, \text{ i.e. } 100 \text{ boxes}$$

Theoretical mean of Poisson distribution $\lambda = 1$

Hence the theoretical frequency for x success is

given by $N.P(x)$

$$\text{Where } x = 0, 1, 2, 3, 4, 5$$

i.e. $400 \cdot \frac{e^{-1}(1)^x}{x!}$ is referred to as theoretical frequency where $x = 0, 1, 2, 3, 4, 5$.

$$= 400 \cdot \frac{e^{-1}(1)^0}{0!}, 400 \cdot \frac{e^{-1}(1)^1}{1!}, 400 \cdot \frac{e^{-1}(1)^2}{2!}, 400 \cdot \frac{e^{-1}(1)^3}{3!},$$

$$400 \cdot \frac{e^{-1}(1)^4}{4!}, 400 \cdot \frac{e^{-1}(1)^5}{5!}$$

$$= 400(e^{-1}), 400(0.36), 200(e^{-1}), 66.67(e^{-1}), 16.67(e^{-1}), 3.33(e^{-1})$$

$$= 400(0.36), 400(0.36), 200(0.36) \dots \quad [Here e^{-1} = 0.36]$$

$$= 147.15, 147.15, 73.58, 24.53, 6.13, 1.23.$$

\therefore The Theoretical frequencies are.

x	0	1	2	3	4	5
f	147	147	74	25	6	1

Normal Distribution \sim

Here we shall consider the continuous distribution namely distribution. A continuous distribution is a

distribution in which a variable can take all values with in a given range Normal distribution is also

known as Gaussian distribution. It is another limiting from of B.D for larger values of n when neither p nor q is very small.

It is derived from B.D by increasing the number of trials indefinitely.

Defn: A random variable 'x' is said to have a normal distribution if its density function (or) probability distribution is given by.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{where } -\infty < \sigma < \infty \\ -\infty < \mu < \infty$$

Where ' μ ' = mean of continuous distribution.

σ = S.D of continuous Distribution.

Constants of Normal distribution :-

$$\text{Mean} :- \mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$\mu = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

no closed form formula exist.

$$\text{Put } z = \frac{x-\mu}{\sigma}$$

$$z\sigma = x - \mu$$

$$x = \mu + z\sigma$$

$$dx = \sigma dz$$

$$= \int_{-\infty}^{\infty} (\mu + z\sigma) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\mu \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz + \sigma \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz \right]$$

either we solve odd & even part separately
else if $\int_{-\infty}^{\infty} f(x) dx = 0$, when $f(x)$ is odd function.

odd part $= \int_{-\infty}^{\infty} f(x) dx = 0$, when $f(x)$ is even function.

$$\text{even part} = \int_{-\infty}^{\infty} f(x) dx = 2 \int_0^{\infty} f(x) dx$$

$$\mu = \frac{2b}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}}$$

mean is sum of $\mu = b$. if x is odd function we get zero

Variance :- $\sigma^2 = E[(x-\mu)^2]$

$$\sigma^2 = E[(x-\mu)^2] = \text{if using definition}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\text{definition} = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\text{solution} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\text{put } \frac{x-\mu}{\sigma} = z. \quad \sigma = \sqrt{\frac{1}{2\pi}} e^{-\frac{z^2}{2}}$$

$$x-\mu = \sigma z$$

$$dx = \sigma dz. \quad \text{outer part} \rightarrow \int_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^2 e^{-\frac{z^2}{2}} \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}} \sigma^2 \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz$$

$$\text{put } \frac{z^2}{2} = t \Rightarrow z^2 = 2t. \quad z = \sqrt{2t}$$

$$dz = \frac{1}{\sqrt{2t}} dt$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} 2t e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} 2t e^{-t} \frac{dt}{\sqrt{t}}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{\frac{3}{2}-1} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma(\frac{3}{2}) = \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} = \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \sqrt{\pi}$$

$$= \sigma^2$$

$$\boxed{\sigma^2 = \sigma^2}$$

Mode :-

Mode is the value of x for which $f(n)$ is max

i.e. mode is the solution of $f'(x) = 0$ & $f''(x) < 0$

$$\text{By defn } f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}} \left(\frac{x-\mu}{\sigma}\right)^2$$

$$f'(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}} \left(\frac{x-\mu}{\sigma}\right)^2 - \frac{1}{\sigma^2} 2 \left(\frac{x-\mu}{\sigma}\right) \left(\frac{1}{\sigma}\right)$$

$$f'(x) = -f(x) \cdot \left(\frac{x-\mu}{\sigma^2}\right)$$

To get max (or) min point equal $f'(x) = 0$. unit-1-Pg-55/c9

$$-f(x) \left(\frac{x-\mu}{\sigma^2} \right) = 0$$

Put $f(x) \neq 0$

$$\Rightarrow \frac{x-\mu}{\sigma^2} = 0 \Rightarrow x-\mu = 0 \Rightarrow x = \mu$$

$$f'(x) = -\frac{1}{\sigma^2} [f(x)(1-0) + (x-\mu)f'(x)]$$

$$= -\frac{1}{\sigma^2} (f(x) + (x-\mu)f'(x))$$

$$= -\frac{1}{\sigma^2} (f(x) + (x-\mu) \left(-\frac{1}{\sigma^2} f(x)(x-\mu) \right))$$

$$= -\frac{f(x)}{\sigma^2} \left[1 - \left(\frac{x-\mu}{\sigma} \right)^2 \right]$$

$$f'(x) = -\frac{f(x)}{\sigma^2} (1-0)$$

$$= -\frac{f(x)}{\sigma^2} < 0 \quad (\because f(x) > 0) \\ \text{(at } x = \mu \text{)}$$

\therefore At $x = \mu$, $f(x)$ has maximum value.

Median \approx

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_{-\infty}^M f(x) dx + \int_M^{\infty} f(x) dx = 1$$

$$\int_{-\infty}^M f(x) dx = \frac{1}{2}$$

$$\int_{-\infty}^M f(x) dx + \int_M^{\infty} f(x) dx = \frac{1}{2}$$

Consider $\int_{-\infty}^M f(x) dx = \int_{-\infty}^M \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}} \left(\frac{x-\mu}{\sigma} \right)^2 dx$.

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^M e^{-\frac{1}{2}} \left(\frac{x-\mu}{\sigma} \right)^2 dx$$

(i) Put $z = \frac{x-\mu}{\sigma} \Rightarrow x-\mu = z\sigma$

$$x = \mu + z\sigma$$

$$dx = \sigma dz$$

Put $x = -\infty$, put $x = \infty$
Hence $\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz = 1$

$$Z = -\infty, Z = \infty$$

Integrating $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz$ with $a = 1$ & $b = \infty$ to obtain

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{z^2}{2}} dz.$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} = \frac{1}{2}$$

Hence Normal distribution is said to be symmetric

$$= \frac{1}{2} + \int_0^m f(x) dx = \frac{1}{2}$$

$$\text{As } \int_a^b f(x) dx = 0 \text{ implies } \left[\frac{f(x)}{x} \right]_a^b = 0 \text{ then } b = a$$

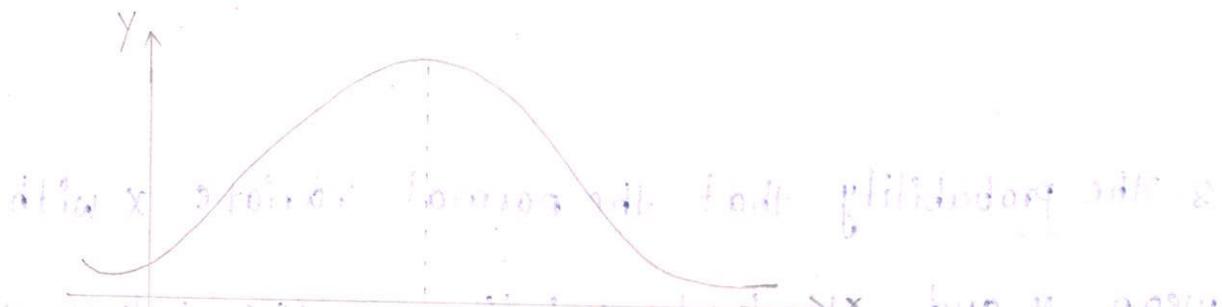
$\therefore m = \mu$

$\therefore \text{Mean} = \text{Mode} = \text{Median}$

\therefore Hence the distribution is Symmetrical.

Characteristic of Normal distribution:

- The graph of the normal distribution $y = f(x)$ in the xy -plane is known as the normal curve.



- The Curve is a bell shaped curve and symmetrical with respect to mean i.e. about the line $x = \mu$ and the two tails on the right and left sides of the mean (μ) extends to infinity. The top of the bell is directly above the mean (μ). Side of the bell is σ .

- Area Under the normal curve represents the total population.

4. Mean, median and mode of the distribution

coincide at $x=\mu$ as the distribution is symmetrical

so, normal curve is unimodal (has only one maximum point).

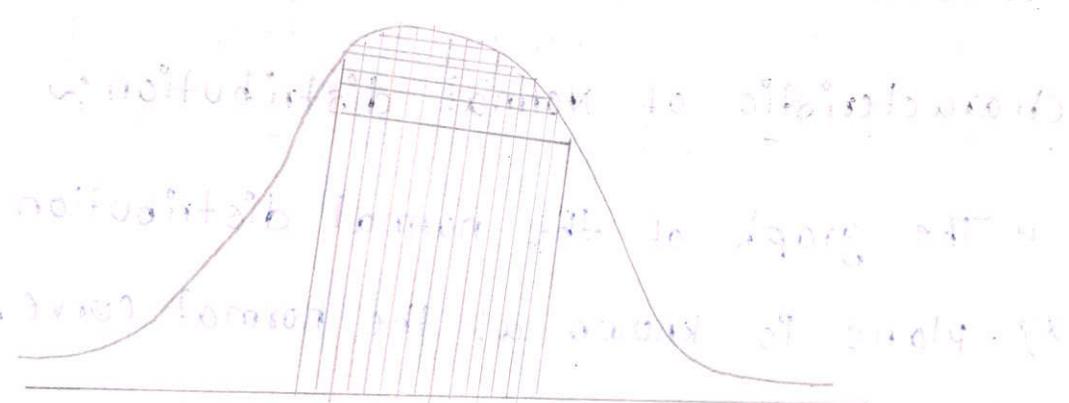
5. x-axis is an asymptote to the curve.

6. Linear combination of independent normal variates is also a normal curve.

7. The points of inflection of the curve are at

$$x = \mu \pm \sigma$$

The curve changes from concave to convex at $x = \mu + \sigma$ to $x = \mu - \sigma$.



8. The probability that the normal variate x with mean μ and standard deviation σ lies between x_1 & x_2 is given by

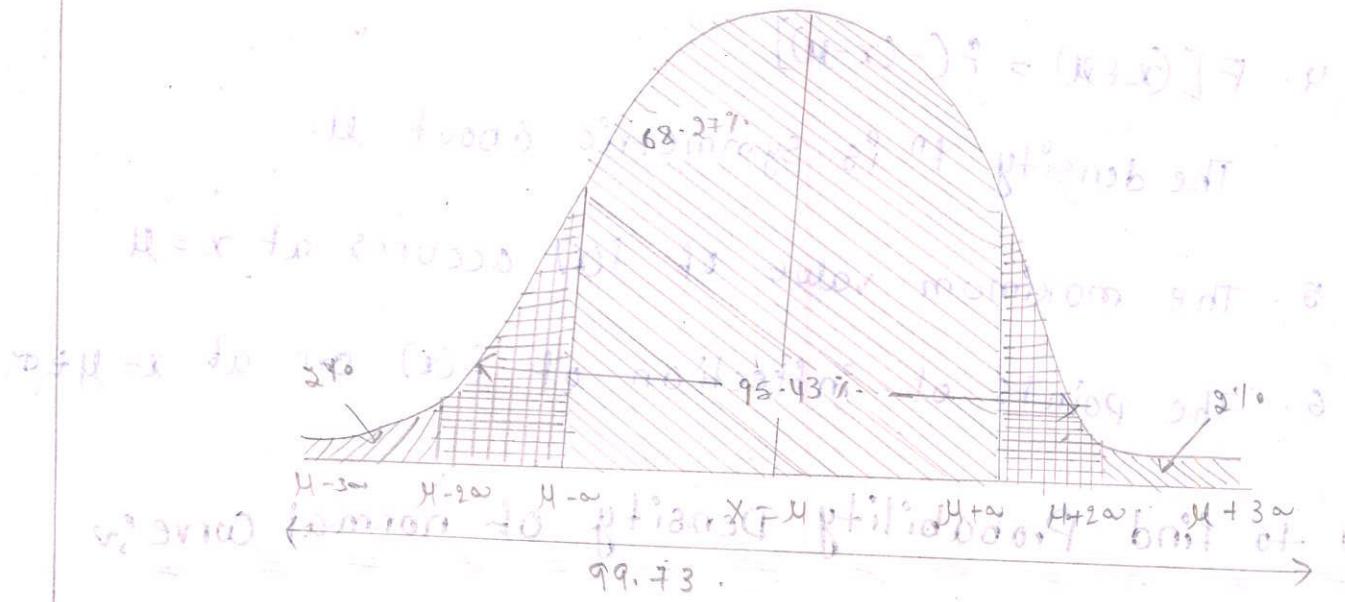
$$P(x_1 \leq x \leq x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad (1)$$

Since (1) depends on the two parameters μ and σ , we get different normal curves for different values of μ and σ . It is impractical task to plot all such normal curves. Instead, we can use standard normal

Instead by putting $z = \frac{x-u}{\alpha}$. The R.H.S of equation (30)

(i) becomes independent of the two parameters μ and σ .
 Here z is known as the standard variable.

9. Area under the normal curve is distributed as follows:-



(i) Area of normal curve between $\mu - \sigma$ and $\mu + \sigma$ is 68.27%
 or mounted 68.27% of the area
 i.e. $P(\mu - \sigma \leq X \leq \mu + \sigma) = 0.6826$, by the normal
 bell-shaped sd and $\sigma \leq x \leq \mu$ is 68.27%.

(ii) Area of normal curve between $\mu - 2\sigma$ and $\mu + 2\sigma$ is 95.43%
 or mounted 95.43% of the area, bracketed 94.53%.

(iii) Area of normal curve between $\mu - 3\sigma$ and $\mu + 3\sigma$ is 99.73%.

Standard normal Distribution

Definition: The normal distribution with mean (μ) = 0 & $\sigma^2 = 1$ is called standard normal distribution.

S.D (σ) = 1 is known as standard normal distribution

The random variable that follows this distribution is denoted by Z . If a variable X follows normal distribution with mean μ and s.d. σ .

The variable z defined as

$$z = \frac{x - y}{\sigma}$$

Properties :-

1. $\int_{-\infty}^{\infty} f(x) dx = 1$
2. $f(x) \geq 0 \forall x$
3. $\lim_{x \rightarrow -\infty} f(x) = 0, \lim_{x \rightarrow \infty} f(x) = 0$
4. $F(x+\mu) = F(-(x-\mu))$
The density f^n is symmetric about μ .
5. The maximum value of $f(x)$ occurs at $x=\mu$
6. The points of inflection of $f(x)$ are at $x=\mu \pm \sigma$.

How to find Probability Density of normal Curve?

The probability that the normal variable x with mean μ and standard deviation σ , lies between two specific values x_1 and x_2 with $x_1 \leq x_2$ can be obtained

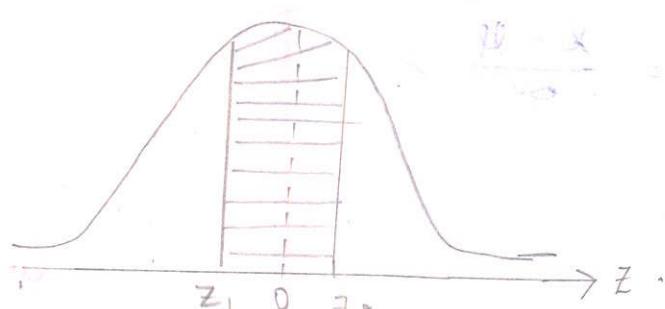
Using area under the standard normal curve as follows:-

Step 1:- Perform the change of scale $z = \frac{x-\mu}{\sigma}$ and find z_1 and z_2 corresponding to the values of x_1 and x_2 respectively.

Step 2(a):- If both $z_1 < 0$ and $z_2 > 0$ are positive, then

$$\text{case (a)}: P(x_1 \leq x \leq x_2) = A(z_2) + A(z_1)$$

$$= P(0 \leq z \leq z_1) + P(0 \leq z \leq z_2)$$



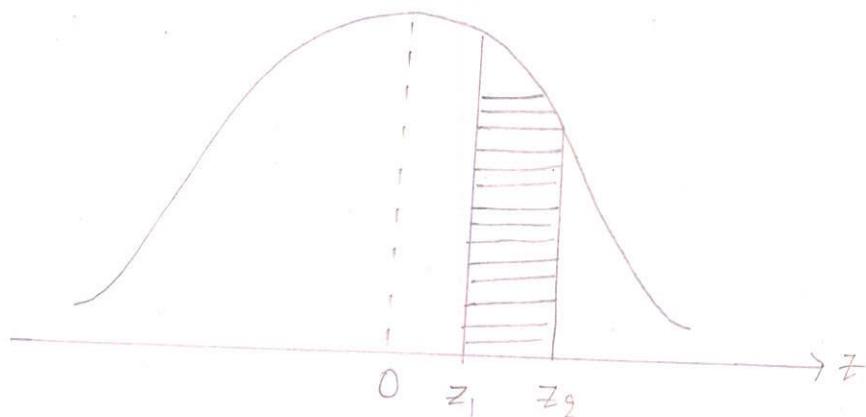
Case 1 :- If both z_1 and z_2 are possible (or both negative).

Then $P(z_1 \leq X \leq z_2) = [A(z_2) - A(z_1)]$

= (Area Under the normal curve from 0 to z_2)

- (Area Under the normal curve from 0 to z_1).

$$P(z_1 \leq X \leq z_2) = P(0 \leq Z \leq z_2) - P(0 \leq Z \leq z_1).$$



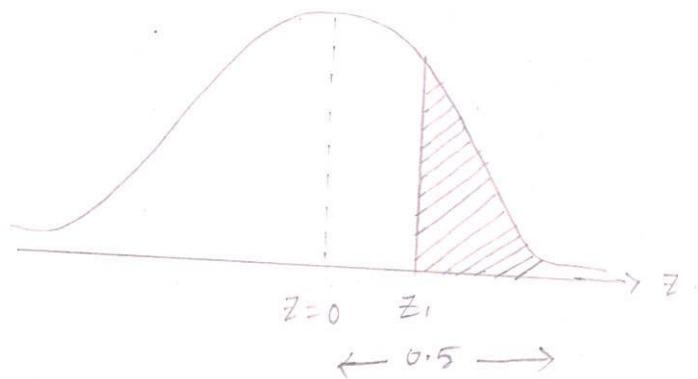
Step 2(b) : To Find $P(Z > z_1)$

case 1 : If $z_1 > 0$ then.

$$P(Z > z_1) = 0.5 - A(z_1)$$

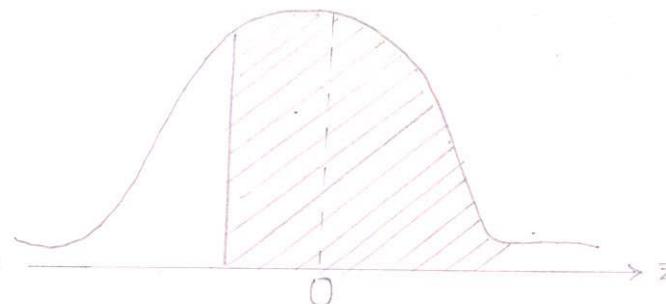
$$\left[\because P(Z < 0) = P(Z > 0) = \frac{1}{2} \right]$$

$$= 0.5 - P(0 \leq Z \leq z_1).$$



Case 2 : If $z_1 < 0$ then $P(Z > z_1) = 0.5 + A(z_1)$

$$= 0.5 + P(0 \leq Z \leq z_1).$$



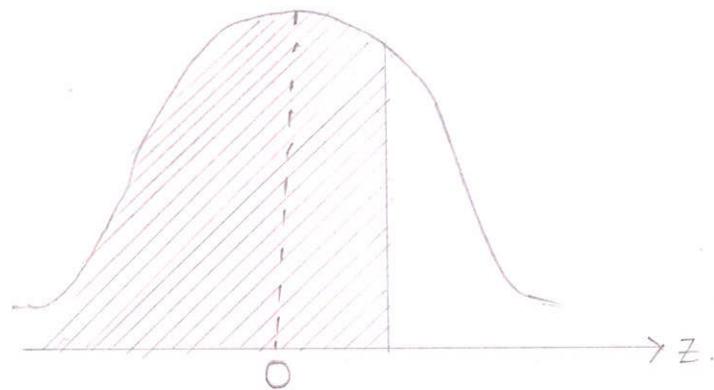
Step 2 (c) :- To find $P(Z < z_1) = 1 - P(Z > z_1)$.

Case 1: If $z_1 > 0$ Then

$$P(Z < z_1) = [1 - P(Z > z_1)] = 1 - [0.5 + A(z_1)]$$

$$\therefore P(Z < z_1) = 0.5 - A(z_1)$$

$$(z_1 > 0) \Rightarrow P[0 \leq Z \leq z_1].$$

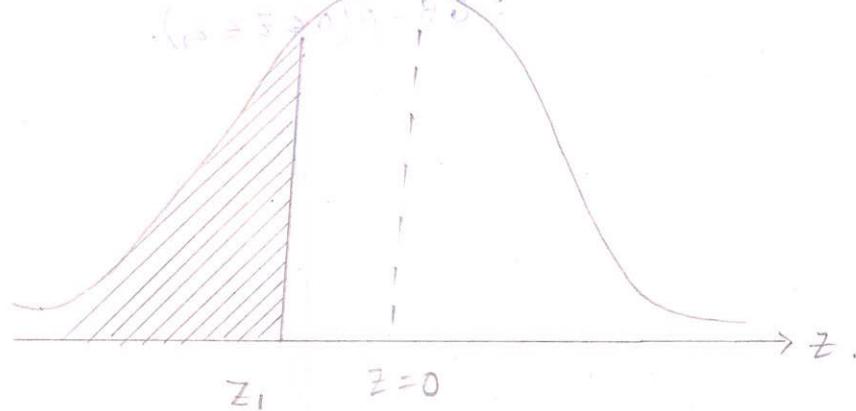


Case 2:- If $z_1 \leq 0$ Then

$$P(Z < z_1) = [1 - P(Z > z_1)] = 1 - [0.5 + A(z_1)]$$

and also 1st case

$$= 0.5 - A(z_1)$$
$$\therefore P(Z < z_1) = 0.5 - P[0 \leq Z \leq z_1].$$



(Ans = 0.5 - (0.5 + 0.05) = 0.5 - 0.55 = 0.45 (Ans))

(Ans = 0.45)

- Q) If x is a normal variate with mean 30 and standard deviation σ . Find the probability that (i) $26 \leq x \leq 40$
(ii) $x \geq 45$.

Soln :- Given mean $\mu = 30$ and S.D $\sigma = 5$

$$(i). \text{ When } x=26, z = \frac{x-\mu}{\sigma} = \frac{26-30}{5} = -0.8 = z_1 \text{ (say)}$$

$$\text{when } x=40, z = \frac{40-30}{5} = 2 = z_2 \text{ (say)}$$

$$\therefore P(26 \leq x \leq 40) = P(-0.8 \leq z \leq 2)$$

$$\begin{aligned} &= A(z_2) + A(z_1) - \\ &= A(2) + A(-0.8) \\ &= 0.4772 + 0.2881 \\ &= 0.7653 \end{aligned}$$

$$(ii) \text{ When } x=45, z = \frac{x-\mu}{\sigma} = \frac{45-30}{5} = 3 = z_1 \text{ (say)}$$

$$P(x \geq 45) = P(z_1 \geq 3)$$

$$= 0.5 - A(z_1)$$

$$= 0.5 - A(3)$$

$$= 0.5 - 0.49865$$

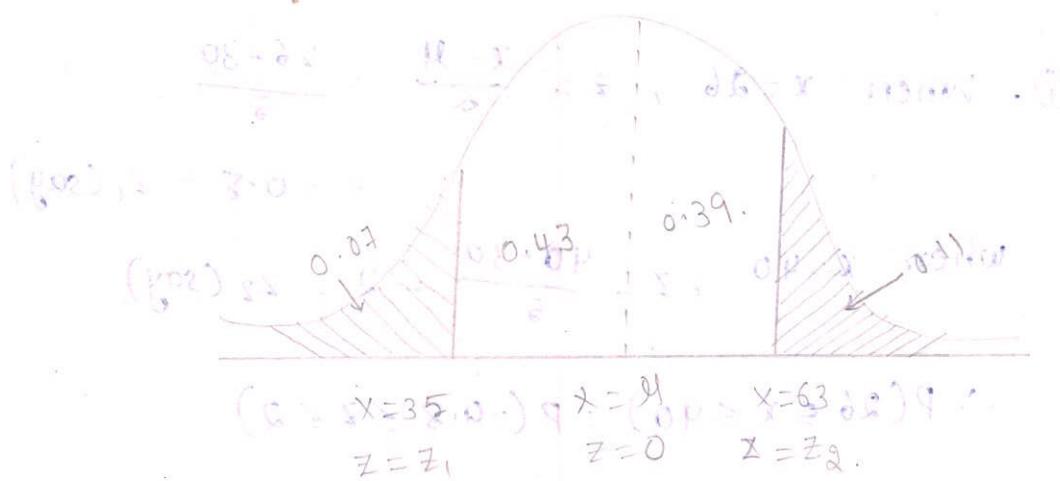
$$= 0.00135.$$

- Q) In a normal distribution : 7% of the items are Under 35 and 89% are Under 63. Determine the mean and variance of the distribution.

Soln :- Let μ be the mean ($z=0$) and σ the standard deviation of the normal curve [7% of the items are Under 35 means the area to the left of the ordinate $x=35$.]

Given $P(X \leq 35) = 0.07$ and $P(X \leq 63) = 0.89$.
 $\therefore P(X \geq 63) = 1 - P(X \leq 63) = 1 - 0.89 = 0.11$

The points $x=35$ and $63=x$ are shown in the fig.



$$\text{When } X=35, z = \frac{x-\mu}{\sigma} = \frac{35-40}{\sigma} = -z_1 \text{ (say)} - ①$$

$$X=63, z = \frac{x-\mu}{\sigma} = \frac{63-40}{\sigma} = z_2 \text{ (say)} - ②$$

From the figure we get

$$P(0 < z < z_2) = 0.89 \Rightarrow z_2 = 1.23 - ③$$

$$P(0 < z < z_1) = 0.43 \Rightarrow z_1 = 1.48 - ④$$

$$\text{From ③, we have } \frac{35-\mu}{\sigma} = -1.48 - ⑤$$

$$\text{From ④, we have } \frac{63-\mu}{\sigma} = 1.23 - ⑥$$

⑥ - ⑤ gives:

$$\frac{63-\mu}{\sigma} - \frac{35-\mu}{\sigma} = 1.23 - (-1.48)$$

$$\frac{28}{\sigma} = 2.71 \quad \text{or} \quad \sigma = \frac{28}{2.71} = 10.332$$

$$\text{From ⑤, we get } 35-\mu = -1.48 \times 10.332 = -15.3$$

$$35-\mu = -15.3 \quad \text{or} \quad \mu = 35 + 15.3$$

$$\text{variance} = \sigma^2 = 106.75$$

(33)

③ In a normal distribution 31% of the items are under 45 and 8% are under 64. Find the mean and variance of the distribution.

Soln :- Let x be the continuous random variable.

Let μ be the mean and σ the standard deviation.

Given $P(x < 45) = 0.31$ and $P(x > 64) = 0.08$

standard variable $z = \frac{x-\mu}{\sigma}$

When $x=45$ Let $z=z_1$

so that $z_1 = \frac{45-\mu}{\sigma} \quad \text{--- (1)}$

$$\therefore \int_{-\infty}^{z_1} \phi(z) dz = 0.31 \quad (\text{or})$$

$$\int_{-\infty}^{\infty} \phi(z) dz - \int_{-\infty}^{z_1} \phi(z) dz = 0.31 \quad \text{--- (2)}$$

$$\therefore \int_{z_1}^{\infty} \phi(z) dz = \int_{-\infty}^{z_1} \phi(z) dz = 0.31 = 0.5 - 0.31 = 0.19. \quad \text{--- (3)}$$

Hence $P(0 < z < z_1) = 0.19$.

$$z_1 = -0.5 \quad (\text{from Table values}) \quad \text{--- (4)}$$

$$\therefore \int_{z_2}^{\infty} \phi(z) dz = 0.08 \quad (\text{or}) \quad \int_0^{\infty} \phi(z) dz - \int_0^{z_2} \phi(z) dz = 0.08$$

$$\text{Hence } \int_0^{\infty} \phi(z) dz = \int_0^{z_2} \phi(z) dz = 0.08 = 0.5 - 0.08 = 0.42.$$

$$\text{Thus } P(0 < z < z_2) = 0.42 \Rightarrow z_2 = 1.4. \quad \text{--- (5)}$$

$$\text{When } x=64, z = \frac{64-\mu}{\sigma} = z_2 \text{ (say)}, \quad \text{--- (6)}$$

$$\text{From (1) \& (6) we have } \frac{45-\mu}{\sigma} = -0.5$$

$$45-\mu = -0.5(\sigma) \quad \text{--- (7)}$$

$$\text{From (3) \& (4) we have } \frac{64-\mu}{\sigma} = 1.4. \quad \text{unit-1-Pg-65/69}$$

$$64 - \mu = 1.4(\sigma) \quad \text{--- (6)}$$

The result of 64 being substituted, we have $\sigma = 3$

⑤ + ⑥ gives $45 + 10 = 45 + 0.5\sigma + 1.4\sigma$

$$(45 - \mu) + (64 - \mu) = 45 + (0.5\sigma) + (1.4\sigma)$$

$$-19 = -1.9\sigma$$

Middle number is 45, so $\sigma = 10$

$$\therefore \sigma = \frac{19}{1.9} = 10.$$

$$\text{From (5)}: \mu = 45 + 10 = 55 \quad \text{and} \quad \text{SD} = 10.$$

Hence mean $= 55$ and SD = 10.

④ The marks obtained in mathematics by 1000 students is normally distributed with mean 78% and SD 11%. Determine

x [i] How many students got obtained by the lowest]

[marks 90%]

(i) How many students got marks above 90%.

(ii) What was the highest mark obtained by the lowest 10% of the students.

(iii), within what limits did the middle of 90% of the students lie.

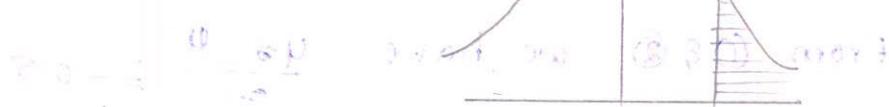
Given mean $\mu = 78$ & SD $\sigma = 11$

(i) When $x = 90$, $z = \frac{x-\mu}{\sigma} = \frac{90-78}{11} = 1.09 = z_1$ (say)

Hence the no. of students with marks more

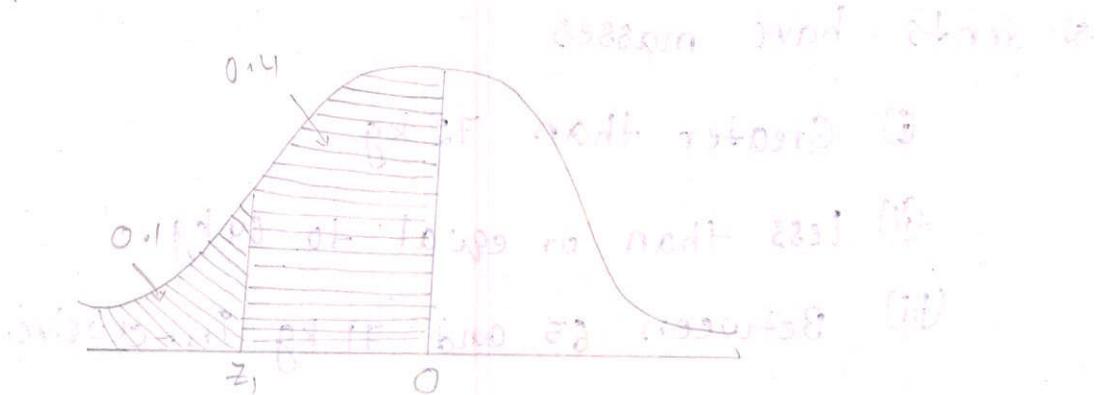
than 90% = $0.1379 \times 1000 = 137.9$ approx

$\therefore 137.9 \approx 138$



$$\therefore P(-1.09 < Z < 1.09) = 0.6826$$

(ii). The 0.1% area to the left of Z corresponds to the lowest 10% of the student.



From figure,

$$0.4 = 0.5 - 0.1 = 0.5 \text{ Area from } 0 \text{ to } Z_1$$

$$\therefore Z_1 = 1.28 \text{ (table).}$$

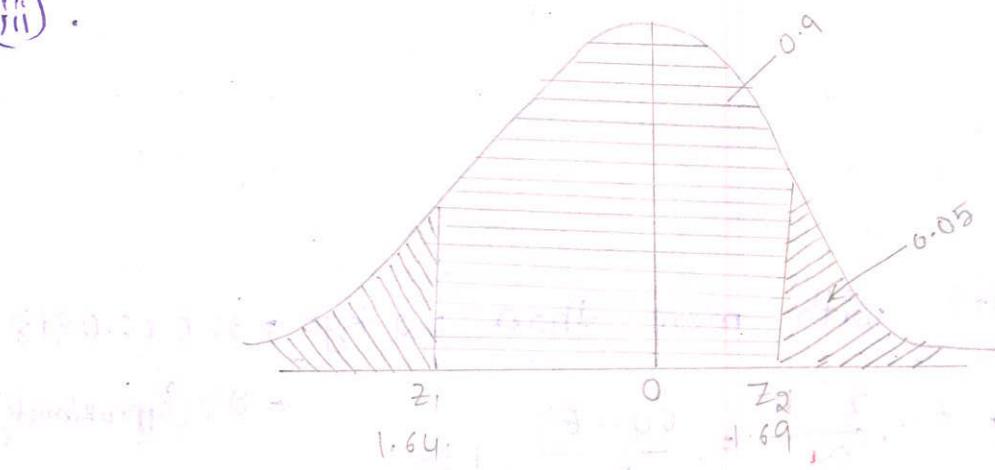
$$\text{Thus } -1.28 = \frac{x - \mu}{\sigma} = \frac{x - 0.78}{0.11}$$

$$x = 0.78 - 1.28 (0.1)$$

$$x = 0.6392$$

Hence the highest mark obtained by the lowest 10% of students $= 0.6392 \times 1000 \approx 64\%$.

(iii) .



Middle 90% correspond to 0.9 area leaving 0.05 area

on both sides. Then the corresponding Z 's are ± 1.64 .

$$\therefore -1.64 = Z_1 = \frac{x_1 - \mu}{\sigma} = \frac{x_1 - 0.78}{0.11}$$

$$x_1 = 0.78 - 1.64 (0.1) = 0.5996 \text{ (or) } 59.96\%$$

$$\text{and } 1.64 = Z_2 = \frac{x_2 - \mu}{\sigma} = \frac{x_2 - 0.78}{0.11}$$

$$x_2 = 1.64 (0.1) + 0.78 = 0.9604 \text{ (or) } 96.04\%$$

Thus the middle 90% have marks in between 59.96 to 96.

If the masses of 300 students are normally distributed with mean 68kgs and s.d 3kgs . how many students have masses

- (i) Greater than 72 kg
- (ii) Less than or equal to 64 kg
- (iii) Between 65 and 71 kg inclusive.

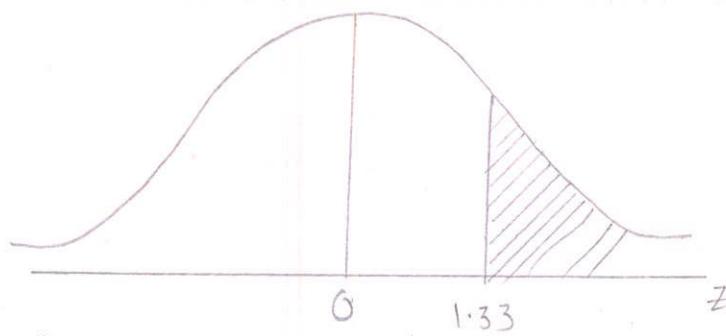
SOL :- Let μ be the mean and σ the standard deviation of the distribution . Then $\mu=68$ kgs & $\sigma=3$ kg

Let the variable x denote the masses of students

$$(i) \text{ When } x=72, z = \frac{x-\mu}{\sigma} = \frac{72-68}{3} = 1.33.$$

$$\therefore P(x>72) = P(z>1.33)$$

$$= 0.5 - A(1.33) = 0.5 - 0.4082 \\ = 0.0918 "$$

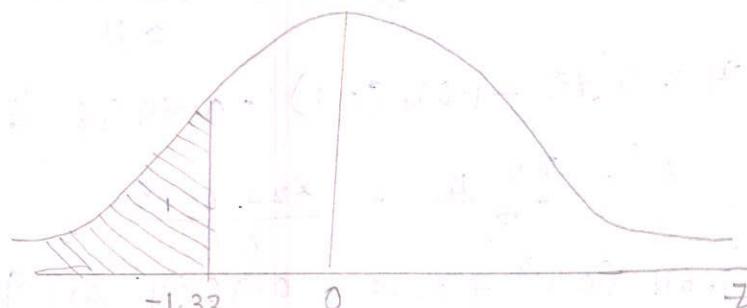


No. of students with more than 72 kgs = 300×0.0918

$$(ii) \text{ When } x=64, z = \frac{x-\mu}{\sigma} = \frac{64-68}{3} = -1.33. \quad = 28 \text{ (approximately)}$$

$$\therefore P(x \leq 64) = P(z \leq -1.33)$$

$$= 0.5 - A(-1.33) = 0.5 - 0.4082 = 0.0918.$$



no. of students have masses less than or equal to 64 kg

$$= 300 \times 0.0918 = 28 \text{ (approximately).}$$

(iii) When $x=65$, $z = \frac{x-\mu}{\sigma} = \frac{65-68}{3} = -1 = z_1$ (say)

$$x=71, z = \frac{x-\mu}{\sigma} = \frac{71-68}{3} = 1 = z_2 \text{ (say)}$$

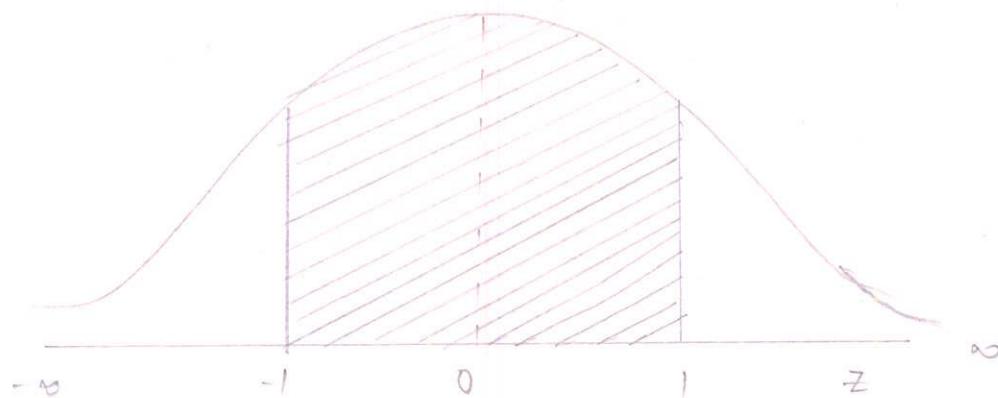
$$\therefore P(65 \leq x \leq 71) = P(-1 \leq z \leq 1)$$

$$= A(z_2) + A(z_1) = A(1) + A(-1)$$

$$= A(1) + A(1)$$

$$= 2A(1)$$

$$= 2(0.3413) = 0.6826.$$



$$\therefore \text{Required number of students} = 300 \times 0.6826$$

$$= 205$$

\approx

